A LAW OF THE INTEGRATED LOGARITHM FOR THE TAIL SUMS OF DYADIC MARTINGALES USING STOPPING TIMES

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Abstract

Stopping times have been used in number of places in the derivation of law of iterated logarithm for various context. In this article, we obtain a law of the iterated logarithm for the tail sums of dyadic martingales using stopping times.

Keywords: Dyadic Martingales, Tail LIL, Stopping times.

1. Introduction

In probability theory, the law of iterated logarithm (LIL) describes the magnitude of the fluctuation of a random walk. Its study is directly or indirectly related to dyadic interval and dyadic martingales. A dyadic interval of the unit cube [0, 1) is of the form $Q_{nj} = \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right)$ for $n, j \in \mathbb{Z}$. Generally, we write $Q_n$ to denote a generic interval of length $\frac{1}{2^n}$ [3]. If $F_n$ denotes the $\sigma$-algebra generated by the dyadic intervals of the form $\left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right)$ on $[0,1)$ then the conditional expectation of $f_{n+1}$ on $F_n$ is given by $E(f_{n+1}|F_n) = \frac{1}{|Q_n|} \int_{Q_n} f_{n+1}(y) dy, \; x \in Q_n$. In this consideration, a dyadic martingale is a sequence of integrable functions $\{f_n\}_{n=0}^\infty$ with $f_n: [0,1) \rightarrow \mathbb{R}$ such that for every $n$, $f_n$ is $F_n$-measurable and $E(f_{n+1}|F_n) = f_n$ for all $n \geq 0$. [2]

For a dyadic martingale, we define the maximal functions as $f^*_m = \sup_{1 \leq k \leq m} |f_k|$ and $f^* = \sup_{1 \leq k < \infty} |f_k|$ and the martingale tail square function is given as $S^2_n f(x) = (S'_n f(x))^2 = \sum_{k=n+1}^\infty d_k^2(x)$, where $d_k = f_k(x) - f_{(k-1)}(x)$ is the general term of martingale difference sequence $\{d_k\}_{1}^\infty$. [2]

In addition, for a dyadic martingale, we have $\{x: f^*(x) < \infty \} = \{x: \lim f_n(x) \text{ exists} \} \text{ a.s.}$ [1]

In this context, a theorem on the tail LIL for dyadic martingales gives an important result which is stated in the following theorem.[4]
Theorem 1 (Tail LIL for Dyadic Martingale)

Let \( \{f_n\}_{n=0}^{\infty} \) be a dyadic martingale. Assume that there exists a constant \( C < \infty \) such that 
\[
\frac{|S_n f(x)|}{|f_n f(y)|} \leq C,
\]
for all \( x, y \in I_n \) where \( I_n = \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right) \) for \( n = 1, 2, 3, \ldots \) and \( j \in \{0, 1, 2, 3, \ldots, 2^n - 1\} \).

Then
\[
\limsup_{n \to \infty} \frac{|f_n(x) - f(x)|}{2S_n f(x) \log \log \frac{1}{S_n f(x)}} \leq 2C \text{ for a.e. } x.
\]

From the assumption, we get \( S f(x) < \infty \) for a.e. \( x \). This shows that the sequence \( \{f_n(x)\} \) converges to \( f(x) \). Thus the tail law of the iterated logarithm gives the rate of convergence of dyadic martingales \( \{f_n\} \) to its limit function \( f \). Moreover, the rate of convergence depends on the tail sums of martingale square function.

As continuation in the tail LIL for dyadic martingales, we obtained a new result which can be considered as the corollary of the theorem on tail LIL for dyadic martingales stated above. Our main result is as follows.

Theorem 2

Let \( \{f_n\}_{n=0}^{\infty} \) be a dyadic martingale. Fix \( \theta \geq 1 \). Define stopping times \( n_k(x) = \min \{n: x \in I_n, \text{ for some } j \in \{1, 2, 3, \ldots, 2^n \} \} \) and \( \forall \ y \in I_n \) \( S_n f(y) < \frac{1}{y^{\theta k}} \). Then for the sequence of stopping times \( n_k(x) \),
\[
\limsup_{k \to \infty} \frac{|f(x) - f_{n_k}(x)|}{\sqrt{2S_n f(x) \log \log \frac{1}{S_n f(x)}}} < \sqrt{3}
\]
for a.e. \( x \).

Proof:

First of all we prove the following estimate for \( \lambda > 0, \eta > 0 \),
\[
|\{x \in [0, 1]: |f(x) - f_n(x)| > \lambda, S_n f(x) < \eta \lambda\}| \leq \exp \left( -\frac{1}{2 \eta^2} \right) \quad (1)
\]
To prove this we have
\[
|\{x: |f(x) - f_n(x)| > \lambda\}| \leq 6 \exp \left( -\frac{-\lambda^2}{2 ||S_n f||_\infty^2} \right)
\]
Here, \( S_n f(x) < \eta \lambda \) gives \( ||S_n f||_\infty^2 \leq \eta^2 \lambda^2 \). So, \( \frac{-1}{||S_n f||_\infty^2} \leq \frac{-1}{\eta^2 \lambda^2} \). So we have,
\[
|\{x \in [0, 1]: |f(x) - f_n(x)| > \lambda, S_n f(x) < \eta \lambda\}| \leq 6 \exp \left( -\frac{-\lambda^2}{2 ||S_n f||_\infty^2} \right)
\]
\[
\leq 6 \exp \left( -\frac{-\lambda^2}{2 \eta^2 \lambda^2} \right)
\]
\[
= \exp \left( -\frac{1}{2 \eta^2} \right)
\]
This is the required result (1).

Now, choose \( \lambda = \frac{(1+\epsilon)\sqrt{2 \log \log \theta^2l}}{\\theta^2} \) and \( \eta = \frac{\theta}{(1+\epsilon)\sqrt{2 \log \log \theta^2l}} \) where \( \theta > 1 \) and \( \epsilon > 0 \). Then using (1) we have,

\[
\left\{ x \in [0, 1]: |f(x) - f_n(x)| > \frac{(1+\epsilon)\sqrt{2 \log \log \theta^2l}}{\\theta^2}, S_n'f(x) < \frac{1}{\theta^{l-1}} \right\}
\leq 6 \exp \left( \frac{-(1+\epsilon)^2(2 \log \log \theta^2l)}{2\theta^2} \right)
= 6 \exp \left( \log (2l \log \theta) \frac{-(1+\epsilon)^2}{\theta^2} \right)
= \frac{6}{(2l \log \theta) \frac{(1+\epsilon)^2}{\theta^2}}
= \frac{6}{(2l \log \theta) \frac{(1+\epsilon)^2}{\theta^2}} \cdot \left( \frac{1}{\theta^2} \right)
\]

Let us choose \( \epsilon = \sqrt{3} \theta - 1 \). Then we have \( \frac{(1+\epsilon)^2}{\theta^2} = 3 \). Thus,

\[
\left\{ x \in [0, 1]: |f(x) - f_n(x)| > \frac{(1+\epsilon)\sqrt{2 \log \log \theta^2l}}{\\theta^2}, S_n'f(x) < \frac{1}{\theta^{l-1}} \right\}
\leq 6 \exp \left( \frac{1}{2(2l \log \theta)} \frac{3}{1^{3}} \right)
= \frac{C}{1^{3}} \text{ (suppose).} \quad (2)
\]

Now, let \( (x) = \sqrt{x \log \log \frac{1}{x}} \). Then \( g(x) \) is an increasing function. So for \( \frac{1}{\theta^2} \leq S_n'f(x) \), we have,

\[
\sqrt{2S_n'^2f(x) \log \log \frac{1}{S_n'^2f(x)}} \geq \sqrt{2 \frac{1}{\theta^2l} \log \log \theta^2l}
\]

Now, using (3), we have,

\[
\left\{ x \in [0, 1]: |f(x) - f_n(x)| > (1 + \epsilon) \sqrt{2S_n'^2f(x) \log \log \frac{1}{S_n'^2f(x)}} \right\}
\leq \sum_{l=k+1}^{\infty} \left\{ x \in [0, 1]: |f(x) - f_n(x)| > (1 + \epsilon) \sqrt{2 \frac{1}{\theta^2l} \log \log \theta^2l}, S_n'f(x) < \frac{1}{\theta^{l-1}} \right\}
\leq \sum_{l=k+1}^{\infty} \left\{ x \in [0, 1]: |f(x) - f_n(x)| > \frac{1}{\theta^l} \sqrt{2 \log \log \theta^2l}, S_n'f(x) < \frac{1}{\theta^{l-1}} \right\}
\]

\[\text{jacem, Vol. 2, 2016} \quad \text{A Law of the Integrated Logarithm for the Tail Sums of Dyadic Martingales Using Stopping Times}\]
\[ \sum_{l=k+1}^{\infty} \frac{1}{l^3} \leq \int_{k}^{\infty} \frac{1}{x^3} \, dx = \left[ -\frac{1}{2x^2} \right]_{k}^{\infty} = \frac{1}{k^2} \]  

So, (4) can be written as,

\[ \left\{ x \in [0, 1) : |f(x) - f_n(x)| > (1 + \epsilon) \sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}} \right\} \leq \frac{C}{k^2} \]

This can be done for every \( n_k(x) \). So summing over all \( k \) we have,

\[ \sum_{k=1}^{\infty} \left\{ x \in [0, 1) : |f(x) - f_n(x)| > (1 + \epsilon) \sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}} \right\} \leq \sum_{k=1}^{\infty} \frac{C}{k^2} = C \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \]

So, by Borel Cantelli lemma, for a.e. \( x \), there exists \( M \) which depends on \( x \) such that for every \( k \geq M \),

\[ |f(x) - f_{n_k}(x)| \leq (1 + \epsilon) \sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}} \]

But we have chosen \( \epsilon = \sqrt{3} \theta - 1 \). So,

\[ |f(x) - f_{n_k}(x)| \leq \sqrt{3} \theta \sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}} \]

that is,

\[ \frac{|f(x) - f_{n_k}(x)|}{\sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}}} \leq \sqrt{3} \theta \]

It is noted that as \( n \to \infty, k \to \infty \). Now, letting \( \theta \downarrow 1 \), we get for a.e. \( x \),

\[ \limsup_{k \to \infty} \frac{|f(x) - f_{n_k}(x)|}{\sqrt{2S_n' f(x) \log \log \frac{1}{S_n' f(x)}}} \leq \sqrt{3} \]
References


