Radiation Coordinates of Florides - McCrea - Synge

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Abstract: In this work we construct the element of volume vector $d\sigma$, of a surface of constant retarded distance around the trajectory of a charged particle with arbitrary motion in a Riemannian space. This constitutes a generalization of the method pioneered by Synge [1] in special relativity. The technique employed is suggested by the ‘radiation coordinates’ $y'$ introduced by Florides-McCrea-Synge [2, 3] in the study of gravitational radiation.

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1. Introduction

Here, the Florides-McCrea-Synge coordinates [2, 3] are used for the electromagnetic radiation and are considerably adapted to this purpose because, for such coordinates, the curved space behaves like a “flat space” in some aspects. That is, the use of $y'$ implies that what was learned in Minkowski space can be naturally translated to a Riemannian space. Our expression for the element of volume vector $d\sigma$, of a surface of constant retarded distance, agrees with that obtained by Villarroel [4] by means of the procedure that DeWitt-Brehme [5] use when constructing a surface with constant instantaneous distance. However, we think that our method is simpler and more powerful, because it turns immediate the results on radiation tensors deduced in [6]. We shall use the World Function $\Omega$ of Ruse [7] which allows having covariant expansions in a curved space. This function remained forgotten for a long time, and its present relevance may be seen in [5, 8-20].

2. Radiation Coordinates

We assume the Dedekind (1868) [21, 22]-Einstein summation convention for the addition of repeated indices, and that the metric locally takes the form, $(\eta_{ab})=(1,1,1,−1)$ at any event. In order to construct the radiation coordinates $y'$ [2] we need a timelike curve $C$ (which in this case will be the electron trajectory) with an orthonormal tetrad on it:

$$\lambda(a), \dot{\lambda}(b) = \eta_{ab}, \quad \lambda(a)\dot{y}^{(a)} = g_{y'j'}, \quad \lambda(a)^{i} = \dot{X}^{i}$$

(1)
where, \( \vec{x'} = \frac{dx'}{ds} \) is the unitary tangent vector to \( C \), and \( x' \) is a totally arbitrary coordinate system with \( ds^2 = g_{ij} \cdot dx^i \cdot dx^j \). The primed indices label points on \( C \). Now let us see how \( x' \) gives new coordinates: We parameterize the null geodesic \( P'P \) in the form \( x'(v) \) with

\[
\frac{dx'}{dv} = \lambda' \quad \text{(retarded point associated to } P) \nonumber
\]

and \( x = x_0 \) at \( P' \), and \( x = x_1 > x_0 \) at \( P \) with \( V' = \frac{dx'}{dv} \) as its tangent vector, satisfying \( V'V = 0 \). The assigned radiation coordinates to \( P \) are given by:

\[
y' = -\Omega_j \lambda^{(r)j} + s \lambda_j \lambda (r) j'
\]

(2)

where \( \Omega_j \) denote the covariant derivative of \( \Omega \), see Synge [14]:

\[
\Omega_j = -(v_1 - v_0) V_j', \quad \Omega_j \Omega^{ji} = 0,
\]

(3)

so that \( y'^\sigma = -\Omega_j \lambda^{(r)j} \), \( y^i = \Omega_j \lambda^j + s \) which implies that in radiation coordinates the curve \( C \) is reduced to \( y'^\sigma = 0, y'^4 = s \). If we introduce the notation:

\[
\xi_j' = -\Omega_j', \quad w = -\xi_j \lambda^j = \Omega_j \lambda^j
\]

(4)

then we obtain the form of the relation (9.3) of Synge [1] for flat space:

\[
y'^\sigma = y'^\sigma = \xi_j \lambda^{(r)j}, \quad y^i = -y^4 = w + s,
\]

(5)

in this sense the curved space behaves like a Minkowski space-time, which is very useful. On the other hand, at \( P' \) the metric tensor can be written in terms of the tetrad as:

\[
g_{ij'} = \lambda^{(r)ij} - \lambda_j \lambda_i
\]

(6)

then \( y'^\sigma y'^\sigma = \xi_i \xi_j (g_{ij'} + \lambda^i \lambda^j') = w^2 \) due to (3, 4), from where \( \xi_j = y'^\sigma \lambda^{(r)ij} + w \lambda^j' \).
therefore $y' - y$ behaves like a null vector $(y' - y')(y, -y') = 0$. Thus, our expressions are compatible with (4, 5, 9) of [1]. Following the corresponding procedure in flat space, let us introduce a new system of coordinates:

$$z^\sigma = y^\sigma, \quad z^4 = y^4 - \sqrt{-g} y^\sigma = s,$$

that is, $z^4$ remains constant on the null cone with vertex at $P'$. It is clear that the Jacobian of the transformation $(y' \longrightarrow z')$ is equal to one, $J (z' / y') = \det (\partial z' / \partial y') = 1$, therefore:

$$J \left( \frac{z^a}{x^b} \right) = J \left( \frac{y^a}{x^b} \right),$$

(8)

now let us calculate (8). We have that

$$\frac{\partial z^\sigma}{\partial x^\nu} = -\Omega_{\nu} \, \mathcal{L}^{(\sigma) | \nu} + N^\sigma \, \Omega_{\nu}, \quad \frac{\partial z^4}{\partial x^\nu} = -w^{-1} \Omega_{\nu}$$

with

$$N^\sigma = w^{-1} \left( \Omega_{\nu} \, \mathcal{L}^{(\sigma) | \nu} + \Omega_{\nu} \frac{d}{ds} \mathcal{L}^{(\sigma) | \nu} + \Omega_{\nu} \frac{d}{ds} \mathcal{L}^{(\sigma) | \nu} \right),$$

where were employed the properties

$$\frac{\partial x^\nu}{\partial x'^{\nu}} = \mathcal{L}^\nu_{\nu}, \quad -w^{-1} \mathcal{L}^\nu_{\nu}, \quad \Omega_{\nu} = (v_i - v_0) \mathcal{V}_i,$$

hence:

$$J \left( \frac{z^a}{x^b} \right) = \mathcal{E}^{lijk} \frac{\partial z^{l}}{\partial x'^{l}} \frac{\partial z^{k}}{\partial x'^{k}} \frac{\partial z^{j}}{\partial x'^{j}} \frac{\partial z^{i}}{\partial x'^{i}} = w^{-1} \mathcal{E}^{lijk} \Omega_{j} \Omega_{j} \Omega_{i} \Omega_{i} \mathcal{L}^{(4) | l} \mathcal{L}^{(2) | k} \mathcal{L}^{(3) | j} \mathcal{L}^{(1) | i},$$

(9)

for the skew-symmetric nature of the Levi-Civita density $\mathcal{E}^{lijk}$. On the other hand, the World Function satisfies $\Omega_{\nu} = \Omega_{\nu}^p \Omega_{\nu}^p$, substituting this into (9) we get:

$$J \left( \frac{z^a}{x^b} \right) = w^{-1} \det (-\Omega_{\nu}^p) \mathcal{E}^{lijk} \mathcal{L}^{(4) | l} \mathcal{L}^{(2) | k} \mathcal{L}^{(3) | j} \mathcal{L}^{(1) | i} \Omega_{\nu}^p;$$

(10)

from (3) it is clear that $\Omega_{\nu}^p$ can be written in terms of the tetrad:

$$\Omega_{\nu}^p = a_{\nu} \mathcal{L}^{(\nu) | \nu} + a_{\nu} \mathcal{L}^{(\nu) | \nu} : w = \Omega_{\nu} \mathcal{L}^{(\nu) | \nu} = -a_{\nu},$$

then, thanks to the skew-symmetry of $\mathcal{E}^{lijk}$, equation (10) acquires the form:

$$J \left( \frac{z^a}{x^b} \right) = \det (-\Omega_{\nu}^p) \mathcal{E}^{lijk} \mathcal{L}^{(4) | l} \mathcal{L}^{(2) | k} \mathcal{L}^{(3) | j} \mathcal{L}^{(1) | i} = \det (-\Omega_{\nu}^p) \det (\mathcal{L}^{(\nu)^p}) = g^{-\frac{1}{2}} (P') D,$$

(11)

where $D = -|\Omega_{\nu}^p|$, $g (P') = -|g_{\nu}|$. Let us introduce the notation:

$$\Delta = g^{-1} D = g^{-\frac{1}{2}} (P) g^{-\frac{1}{2}} (P') D, \quad g (P) = -|g_{\nu}|,$$

(12)

thus from (11):

$$J \left( \frac{z^a}{x^b} \right) = -g^{-\frac{1}{2}} (P) \Delta.$$
Taking into account the last identity it is clear the remark in [5] page 231 and [10] page 1251: the geodesics emerging from \( P \) begin their intersection when \( \Delta = 0 \), arising the so-called ‘caustic surface’. We shall therefore accept that \( P \) is near to \( P' \), in order to have this only geodesic between them. The analysis performed allows consider the volume element of the curved space-time:

\[
d^4x = \left| J \left( \frac{x}{z} \right) \right| d^4z = g^{-\frac{1}{2}}(P) \Delta^{-1} dsd^3z, \tag{14}
\]

but \( z'^a = wp'^a = wp, \mathcal{A}^a(\sigma) \) with \( p_i = w^{-1} \xi_i - \lambda_i \) = unitary spacelike vector:

![Fig. 2. The quantities \( p'^a \) represent the components of \( p' \) in the basis \( \mathcal{A}^a(\sigma) \).](image)

Therefore, \( z^1 = w \sin \theta \cos \phi \), \( z^2 = w \sin \theta \sin \phi \), \( z^3 = w \cos \theta \) which implies \( d^3z = w^2 d\omega d\gamma \) where \( d\gamma = \sin \theta d\theta d\phi \) is the element of solid angle in the rest frame of the charge. Then (14) adopts the form:

\[
d^4x = g^{-\frac{1}{2}}(P) \Delta^{-1} w^2 dsd^3\omega d\gamma, \tag{15}
\]

which together with (13) represents the generalization to Riemannian spaces of the results (9.15, 21) of Synge [1] (who made use of imaginary coordinates) for Minkowski space-time:

\[
J \left( \frac{z^a}{x^a} \right) = -1, \quad d^4x = w^2 dsd^3\omega d\gamma. \tag{16}
\]

In the next section we will apply (15) to the particular case of the surface \( w = \text{constant} \), which is important when studying the electromagnetic radiation

3. Surface of Constant Retarded Distance

Let us consider the 3-space \( w = \text{constant} \), then the covariant derivative \( w_r \) is orthogonal to that surface. It is therefore evident that its vector volume element is given by (where \( d\sigma \) is the 3-element of volumen):

\[
d\sigma = \left| w_{,a}w^a \right|^{-\frac{1}{2}} w_r d\sigma. \tag{17}
\]
But when building the shell formed by $w, w + dw$ and the null cones at $P_1$ and $P_2$, we get for its 4-volume $d^4x = dw \, d\sigma = \left| w, w + \frac{\sqrt{2}}{2}\right| \cdot dw \cdot d\sigma$, and after comparison with (15) implies that $\sqrt{2} d\sigma = g^{-\frac{1}{2}} \left( P \right) \Delta^{-1} w^2 \, ds \, d\gamma$, then (17) acquires the following form:

$$d\sigma_r = g^{-\frac{1}{2}} \left( P \right) \Delta^{-1} w^2 \, w_r \, ds \, d\gamma.$$  

(18)

On the other hand, from (4) we deduce the expression:

$$w_r = \Omega_r, \lambda - w^{-1} \left( \Omega_r \lambda + \Omega_r \frac{d}{ds} \lambda \right) \Omega_r = \dot{\omega}_r - w^{-1} (X + W) \Omega_r,$$  

(19)

where we used the notation $\dot{\omega}_r = \Omega_r, \lambda^r, X = \Omega_r, \lambda^r \lambda^r, W = \Omega_r \frac{d}{ds} \lambda^r = \Omega_r \mu^r$.

The substitution of (19) into (18) provides the result (3.35) of [4]:

$$d\sigma_r = g^{-\frac{1}{2}} \left( P \right) \Delta^{-1} w \left[ \dot{w} - w^{-1} (X + W) \Omega_r \right] \, ds \, d\gamma.$$  

(20)

which is the generalization to curved spaces of the result (10.6) in [1]. The deduction of (20) was simple thanks to the radiation coordinates. Nevertheless, the usefulness of $\dot{z}$ goes far beyond that; in our opinion, its true importance lies on the analogies that we can establish with the Minkowski space-time, which will be seen more clearly in the next section.

4. Radiation Tensors

In a flat space we have the following radiative part of the Maxwell tensor corresponding to the Liénard-Wiechert retarded field [23]:

$$\frac{T_{\mu \nu}^r}{R} = e^2 \, w^{-1} \left( \mu^2 - w^{-2} W^2 \right) \xi_{\mu} \xi_{\nu}, \quad e' = \frac{e}{4\pi},$$  

(21)
with $\mu^2 = \mu, \mu', \mu_r = \frac{d\lambda}{ds}, w = -\xi, \mu' \cdot W = -\xi' \cdot \lambda$, which satisfies:

$$T_{rs}^s \xi^s = 0, \tag{22}$$

$$T_{rr}^s \xi^s = 0. \tag{23}$$

A tensor field is said to be of the radiative type when it satisfies the properties (22) and (23). The continuity equation (23) is consequence of:

$$0 = s_{rs} R \xi^s, \quad (22)$$

$$0 = s_{rs} R . \quad (23)$$

which in turn are particular cases of the identity:

$$\left( f \left( \mu^2 W^m \xi^s \xi^t \right) \right) = 0, \quad -n - m = -4, \tag{25}$$

$f$ being an arbitrary function of $\mu^2$. It seems natural to wonder whether (21) can be extended to the curved space. The answer is positive under the two following prescriptions:

a).- Identify $\xi$ with $-\Omega$, see (4).

b).- Multiply (21) by $\left[ J \left( \frac{z^0}{x^0} \right) \right] = g^{\frac{1}{2}} (P) \Delta$ due to the fact that $d^4x$ contains the factor $g^{\frac{1}{2}} (P) \Delta^{-1}$ with respect to the corresponding expression for the flat space, see (16).

Thus

$$T_{rs} = e^{-2} g^{\frac{1}{2}} (P) \Delta w^{-4} (\mu^2 - w^2 W^2) \Omega_r \Omega_s \tag{26}$$

satisfies (23) with covariant derivative, due to the fact that the validity of (22) turns out to be evident. We can also expect the generalization of (24):

$$\left[ g^{\frac{1}{2}} (P) \Delta \mu' W^4 \Omega_r \Omega_s \right] = 0, \quad \left[ g^{\frac{1}{2}} (P) \Delta w^4 W^2 \Omega_r \Omega_s \right] = 0, \tag{27}$$

besides from (15) and (26) we have:

$$\frac{R}{d^4x} = e^{-2} w^{-2} (\mu^2 - w^2 W^2) \xi^s \xi^t ds dt d\gamma \tag{28}$$

which is important when performing some integrations around the world line of the charged particle. It is worth noting that (26) and (27) correspond to the results (2.28,...,31) of Villarroel [6]. However, in our approach they can be obtained in a natural way by means of an explicit correspondence with the Minkowski space-time. The verification of (27) can be found in the work of the aforementioned author.
References

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