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LAVI-CIVITA CONNECTION OR RIEMANNIAN CONNECTION

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Abstract

The Present paper deals with a connection on a Riemannian manifold M with the help of Riemannian metric.

Key words

Riemannian manifold; affine connection; metric connection (Compatible); Riemannian connection

Introduction

It has been seen that at each point of differentiable Manifold M that is an n-dimensional tangent space T_p(M. hence the tangent spaces $T_p(M)$ and $T_q(M)$ at points p and q are isomorphic, since they are both n-dimensional. However in order to obtain a definite isomorphism relating T_p and T_q it is necessary to introduce some additional structures on the Manifold called a connation. This connects the tangent spaces at different points on the Manifold.

An affine connection is a geometric object on a smooth Manifold which connects nearby tangent spaces, so it permits tangent vector fields to be differentiated as if they were functions on the Manifold with values in a fixed vector spaces.

Yano (1970) studied semi-symmetric metric connection in Riemannian manifold, Sharfuddin and Husain (1976) studied Semi-symmetric connection in almost contact manifold. Golab (1976) defined semi-symmetric and quarter symmetric linear connection and derived their properties.

In this paper Levi-Civita connection or Riemannian is defined with the help of affine connection and it is proved that every Riemannian manifold admits a unique Riemannian connection (De, 2009).

1. Preliminaries

A smooth manifold M with a Riemannian metric g (a covariant tensor field of degree 2 i.e. of type (0,2) is called a Riemannian manifold if it satisfies the following conditions.

i) g is symmetric, i.e. g(X,Y,) = g(Y,X) for all X,Y$\varepsilon\chi$ (M)

ii) g is positive definite, i.e. g(X,X)≥0 for all Xεχ(M) and g(X,X)=0 iff X=0

A Riemannian manifold is denoted by (Mn,g) or (M,g) or simplify by M.

Definition 2.1 Affine connection or linear connection

An affine connection on a manifold M is a bilinear mapping.

$\nabla: \chi(M)\times \chi(M) \rightarrow \chi(M)$

Written

$\nabla(X,Y) = \nabla_X Y$ *or* $\nabla_X (Y)$ and satisfies

the following properties.

i) $\nabla \alpha_X + \beta_Y Z = \alpha\nabla_X Z + \beta\nabla_Y Z,$

ii) $\nabla f_X + g Y Z = f\nabla_X Z + g\nabla_Y Z$

iii) $\nabla_X(f_Y + gZ) = f\nabla_X Y + (Xf)Y + g\nabla_X Z + (Xg)Z$

for all $\alpha, \beta, \in R, f,g\ C^\infty$ and $X,Y,Z \in \chi(M)$

An affine connection is also called a linear connection a Koszul connection. The operator ∇x is called the covariant differentiation with respect to X and the vector field ∇xYεχ(M) is called the covariant derivative of Y with respect to X.

Definition 2.2

An affine connection ∇ on an n-dimensional Riemannian manifold (M,g) is called a metric Compatible connection or simplify metric connection if

$\nabla g=0$

i.e. $(\nabla_X g)(Y,Z) = 0$

for all $X,Y,Z \in \chi(M)$

Definition. 2.3 Levi-civita connection or Riemannian connection.

Let (M,g) be a Riemannian manifold of dimension n with an affine connection ∇. Then the affine connection ∇on M is said to be Levi- Civita connection or Riemannian connection if, it satisfies the following.

i) ∇is symmetric or torsion free

i.e. ∇XY–∇Y X = [X,Y] and

ii) ∇ is a metric connection

i.e. (∇*x* g) (Y,Z) =0 for all x,y,zεχ(M)

Thus a Riemannian connection or Levi-Civita connection on a Riemannian manifold is a linear connection which is torsion free and metric compatible.

3. Theorem (Fundamental Theorem of Riemannian geometry)

Every Riemannian manifold (M,g) of dimension n admits a unique torsion-free metric connection.

Proof

To prove this theorem, first we define a connection ∇ on the Riemannian manifold (M,g) as a mapping

∇: χ(M)xχ(M)→ χ(M)

such that ∇ satisfies the relation

2g(∇xYZ)= Xg(Y,Z) +Yg(Z,X)-Zg(X,Y) + g([X,Y],Z) -g([Y,Z], X) +g([Z,X],Y)

For all X,Y,Z εχ (M). Then we shall show that such a connection ∇ defined above is a linear connection, metric compatible, Vanishing torsion tensor and unique

First Let us verify that ∇ is a linear connection on M.

Since for all X,Y,Z εχ(M) , Uεχ (M and α, βεR we have by definition.

$$2g(\nabla_X(\alpha Y+\beta Z),U)=Xg(\alpha Y+\beta Z,U)+(\alpha Y+\beta Z)g(U,X)-Ug(X,\alpha Y+\beta Z)+g([X,\alpha Y+\beta Z],U)$$

$$-g([\alpha Y+\beta Z,U],X)+g([U,X],\alpha Y+\beta Z)$$

$$=\alpha\{Xg(Y,U)+Yg(U,X)-Ug(X,Y)+g([X,Y],U)-g([Y,U],X)+g([U,X],Y)\}+$$

$$\beta\{Xg(Z,U)+Zg(U,xX)-Ug(X,Z)+g([X,Z],U)-g([Z,U]),X+g((U,Z),Z)\}$$

$$=2\alpha g(\nabla_X Y,U)+2\beta g(\nabla_X Z,U)$$

i.e. $g(\nabla_X(\alpha Y+\beta Z)-\alpha\nabla_X Y-\beta\nabla_X Z,U)=0$

since U is an arbitrary vector field on M, and g is non- degenerate, it following that

$\nabla_X(\alpha Y+\beta Z)-\alpha\nabla_X Y-\beta\nabla_X Z=$ (i)

i.e. $\nabla_X(\alpha Y+\beta Z)=\alpha\nabla_X Y+\beta\nabla_X Z$ for all $X,Y,Z\in\chi(M)$.

Similarly, we can show that

$\nabla_{\alpha X+\beta Y}Z=\alpha\nabla_X Z+\beta\nabla_Y Z=$ (ii)

For all $\alpha,\ \beta\in R$ and $X,Y,Z\in\chi(M)$.

Thus (i) and (ii) implies that ∇ is R-bilinear.

Again for all $f.h\in C^{\infty}(M)$ and $X,Y,Z,\ U\in\chi(M)$ we have by definition.

$2g(\nabla_{fX+hY}Z,\ U)=$

$(fX+hY)g(Z,U)+Zg(U,FX+hY)-Ug(fX+hY,Z)+g([fX+hY,Z],U)-$

$g([Z,U],fX+hY)+g([U,fX+hY],Z)$

$=fXg(Z,U)+hYg(Z,U)+(Zf)g(U,X)+fZg(U,X)+(Zh)$ $g(U,Y)+hZg(U,Y)-(Uf)\ g(X,Z)-fUg(X,Z)-(Uh)$ $g(Y,Z)-hUg(Y,Z)+fg([X,Z],U)-(Zf)\ g(X,U)+hg$ $([Y,Z],U)-(Zh)\ g(Y,U)-fg\ ([Z,U],X)-hg\ ([Z,U],\ Y)+fg$ $([U,X],Z)+(Uf)\ g(X,Z)+hg([U,Y]),\ Z)+(Uh)g(Y,Z)$

Since $[fX,Y]=f[X,Y]-(Yf)X$

And

$[X,fY]=f[X,Y]+(Xf)Y$ and g is $C^{\infty}(M)$ bilinear

$=f\{Xg(Z,U)+Zg\ (U,X)-Ug(X,Z)+g([X,Z],U)-g$ $([Z,U],\ X)+g\ ([U,X],\ Z)\}+h\{Yg\ (Z,U)+Zg\ (U,Y)-$ $Ug(Y,Z)+g([Y,Z],U)-g([Z,U],Y)+g([U,Y],Z)\}$

$=f2g(\nabla_X Z,U)+h2g\ (\nabla_Y Z,U)$

$=2g\ (f\nabla_X Z,U)+2g\ (h\nabla_Y Z,U)$ since g is $C^{\infty}(M)$ bilinear

i.e. $g\ (\nabla_{fX+hY}Z-f\nabla_X Z-h\nabla_Y Z,U)=0$

Since this is true for any $U\in\chi(M)$ and g is a non-degenerate , it follow that

$\nabla_{fX+hY}Z=f\nabla_X Z+h\nabla_Y Z$ (iii)

For all $X,Y,Z\in\chi(M)$ and $f,h\in C^{\infty}(M)$

Now for all $f,h\in C^{\infty}(M)$ and for all $X,Y,Z,\ U\in\chi(M)$ we get by definition

$2g\ (\nabla_X(fY+hZ),U)=Xg\ (fY+hZ,\ U)+(fY+hZ)\ g(U,X)$ $-Ug\ (X,\ fY+hZ)+g([X,fY+hZ)],\ U)-g([fY+hZ,\ U],X)$ $+g([U,X],\ fY+hZ)$

$=(Xf)\ g(Y,U)+fXg(Y,U)+hXg\ (Z,U)+(Xh)g(Z,U)$ $+fYg(U,X)+hZg(U,X)-fUg(X,Y)-(Uf)\ g(X,Y)-$ $hUg(X,Z)-(Uh)\ g(X,Z)+fg([X,Y]),\ U)$ $+(Xf)g(Y,U)+hg([X,Z],U)+(Xh)g(Z,U)-fg([Y,U],X)$ $+(Uf)g\ (Y,X)-hg\ ([Z,U],\ X)+(Uh)g\ (Z,X)+fg\ ([U,X],$ $Y)+hg\ ([U,X]),Z)$

$=f\ Xg(Y,U)+Yg(U,X)-Ug(X,Y)+g([X,Y],U)-$ $g([Y,U],X)+g\ ([U,X),Y]\}+h\{Xg\ (Z,U)+Zg(U,X)-$ $Ug(X,Z)+g([X,Z],U)-g([Z,U]),X)$ $+g([U,X],Z\}+2(Xf)g(Y,U)+2\ (Xh)\ g(Z,U)$

$=f2g(\nabla x Y,U)+h2g(\nabla x\ Z,U)+2(Xf)\ g(Y,U)+2(Xh)$ $g(Z,U)$

$=2g(f\nabla xY,U)+2g\ (h\nabla xZ,U)+2g((Xf)$ $Y,U)+2g((Xh)Z,U)$

$i.e,\ g(\nabla_X(fY+hZ)-f\nabla_X Y-h\nabla_X Z-(Xf),\ Y-(Xh)Z,\ U)=0$

Since this is true for any $U\in\chi(M)$ and g is non-degenerate, it follows that

$\nabla_X(fY+hZ)-f\nabla_X Y-h\nabla_X Z-(Xf)\ Y-(Xh)Z=0$

i.e, $\nabla_X(fY+hZ)=f\ \nabla_X Y+h\nabla_X Z+(Xf)Y+(Xh)Z$ (iv)

for al $X,Y,Z\in\chi(M)$, $f,h\in C^{\infty}(M)$

Thus from (i) and (iv) , it follows that ∇ is a linear connection on M.

Next we verify the torsion tensor T(X,Y) of this linear connection∇ Vanishes identical.

By definition, we how.

$2g(\nabla_Y X, Z) = Yg(X,Z) + Xg(Z,Y) - Zg(Y,X) + g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)$

Since $[X,Y] = -[Y,X]$ for all $X,Y \in \chi(M)$ and g is symmetric, it follows that

$2g(\nabla_X Y, Z) - 2g(\nabla_Y X, Z) = 2g([X,Y],Z)$

i.e. $g(\nabla_X Y - \nabla_Y X - [X,Y], Z) = 0$

Since g is non- degenerate this implies that

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0 \qquad (v)$$

i.e. $T(X,Y) =$ for all X,Y

Now we show that ∇ is a metric connection.

By definition of ∇, we get by Virtue of skew symmetry of Lie- bracket and symmetry of g that

$2g(\nabla_X Y, Z) + 2g(\nabla_X Z, Y) = 2Xg(Y,Z)$

i.e. $Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$

$$\text{i.e. } (\nabla_X g)(Y,Z) = 0 \qquad (vi)$$

for all $X,Y,Z \in \chi(M)$. Thus ∇ is metric connection

Finally we prove that $\nabla$is unique.

For this, if possible, suppose that M admits, another connection$\overline{\nabla}$ satisfying properties (i)-(vi) i.e. $\overline{\nabla}$ is another torsion free metric connection on M. Then we have

$\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = 0$ and $(\overline{\nabla}_X g)(Y,Z) = 0$ for All $X,Y,Z \in \chi(M)$

Also we have by definition.

$$2g(\overline{\nabla}_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

Hence $2g(\nabla_X Y, Z) - 2g(\overline{\nabla}_X Y, Z) = 0$

i.e, $g(\nabla_X Y - \overline{\nabla}_X Y, Z) = 0$

Since this is true for all $Z \in \chi(M)$ and g is non-degenerate, it follows that

$\nabla_X Y - \overline{\nabla}_X Y = 0$

$$\text{i.e. } \nabla_X Y = \overline{\nabla}_X Y \qquad (vii)$$

for all $X,Y \in \chi(M)$

This shows that the connection ∇ is unique. Thus by Virtue of (i)–(vii)

It implies that every Riemannian manifold (M,g) admits a unique Levi-Civita or Riemannian connection.

Conclusion

For any smooth manifold M with a smooth Riemannian metric g there exists a unique Riemannian connection ∇ on M corresponding to g.

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