Academic Voices
A Multidisciplinary Journal
Volume 2, No. 1, 2012

TOPOLOGICAL GAMES ON PRODUCT SPACES AND ITS FUZZIFICATION

Bidyanand Prasad
Department of Mathematics, TU
TRM Campus, Birganj

B.P. Kumar
University Dept. of Mathematics
B.R.A.B.University, Muzaffarpur, Bihar

Abstract

This paper is concerned with the introduction of an infinite positional game of pursuit and evasion over an ideal of a topological space. A topological game has been played over some new D-product and C-product spaces of two Hausdorff topological spaces. Perfect information, decisions and goals in a game may not be feasible. Hence, fuzzy set theory has been applied in this paper to obtain better results.

Keywords

Topological games, product spaces, fuzzification, coalition, feasible solution.

Introduction

Over an ideal of a topological space, Kumar (1982) has played a topological game. By introducing the concept of rectangle in a topological product spaces, some special types of product are studied. A game is played over such products. It is explained how fuzzy set theory can be applied to obtain better results lastly.

Games over an ideal of a topological space

Let G (I, X) be an infinite positional game of pursuit and evason over I where X is a topological space and $I \subset \subset P(X)$ s.t. I is closed with respect to union and I possesses hereditary property. Such collection I is called an ideal over X.

This game is played as follows: There are two players-P (Pursuer) and E (Evader), They choose alternately consecutive terms of a sequence <En / n ∈ N, where N = {0, 1, 2,........... n, ...} > of subsets of X s.t. each player knows I, E_o, E_1,.......,En when he is choosing En+1.

Sequence <En> of subset of X is said to be a play of the game if for all n ∈ N the following holds:

i. $E_o = X$

ii. E_1, E_3, E_5,..........,E_{2+1} are the choice of P.

iii. E_1, E_3, E_5................E_{2n+1} ∈ I.

iv. E_2, E_4, E_6.................E_{2n+2} are the choice of E.

v. $E_1, E_2, \subset\subset E_o; E_3, E_4 \subset\subset E_2$..........; $E_{2n+1}, E_{2n+2} \subset\subset E_{2n}$.

vi. $E_1 \cap\cap E_2 = \phi, E_3 \cap\cap E_4 = \phi$, ..., $E_{2n+1} \cap\cap E_{2n+2} = \phi$.

If $\cap\cap$ $<E_{2n}> = \phi$ then player P wins the play, otherwise Evader wins the play.

A finite sequence $<E_m / m \leq n>$ is admissible for the game if the sequence $<E_o, E_1$, ...E_n, ϕ, ϕ, ϕ, ...$\phi>$ is a play of the game. For admissible sequence $<E_o$,......,$E_n>$ and even n if s : $<E_o$,.......$E_n> \rightarrow P(X)$ and s ($<E_o$,.......$E_n>$) = E_{n+1} then s is a strategy for player P.S is said to be strategy for evader E if n is odd.

A strategy s is said to be winning for player P in the game G(I,X) if P wins each play of the game with the help of this s. Similarly s is said to be winning for E if E wins each play of the game with the help of s. We denote by P(I, X), the set of all winning strategies of P in the game G(I,X) and by E (I,X), the set of all winning strategies of E in the game G (I,X).

A topological space X is said to be I-like if the set of all winning strategies of player P is not empty i.e. if P (I, X) ≠ ϕ.

Similarly, a space X is said to be anti I-like if the set of all winning strategies of player E is not empty. That is E (I, X) ≠ ϕ

The game G(I, X) is said to be determined, if P (I, X) ≠ φ or E (I, X) ≠ φ

Products of topological spaces

A subset A x B of a topological product space X x Y is called a rectangle. A rectangle E is said to be:

i. Cozero if E' & E" are cozero in X x Y ;

ii. Zero if E' & E" are zero in X x Y ;

iii. Open if E' & E" are open in X x Y ;

iv. Closed if E' & E" are closed in X x Y

where E' & E" are the projections of E into X and Y respectively so that E = E' x E".

A topological product X x Y is said to be strong rectangular if each locally finite open cover of X x Y has a locally finite refinement by cozero rectangles.

The following conditions are seen to be equivalent:

i. The product X x Y is strongly rectangular.

ii. Each finite open cover of X x Y has a locally finite refinement by cozero rectangles.

iii. For each closed subset F and each open set U of X x Y with F $\subset\subset$ U, there is a locally finite collection W by cozero rectangles s.t. F $\subset\subset \cup$ W $\subset\subset$ U

iv. X x Y is normal and for each zero-set F and each cozero-set U of X x Y with F $\subset\subset$ U, there is a locally finite collection W by cozero rectangles such that F W U.

v. There exists a continuous map

f : X x Y $\rightarrow$ [o, 1] such that f (x, y) = $\sum g_t(x) h_t(y)$,t ∈ Twhere g_t : x $\rightarrow$ [o,1] and h_t : Y $\rightarrow$ [o,1] are continuous.

Games over spaces

Each topological space considered in this paper is assumed to be a Hausdorff space. N denotes the set of all natural numbers and m denotes an infinite cardinal number. Also let L = {E_1 ¦ E_1 are closed subsets of X}.

There are two players P and E. Player P chooses a closed set E_1 of X with $E_1 \in L$ and player E chooses an open set U_1 of X with E_1 U_1.

Again, player P chooses a closed set E_2 of X with $E_2 \in L$ and player E chooses an open set U_2 of X with E_2 U_2 and so on.

The infinite sequence $<E_1, U_1, E_2, U_2$,......> is a play of G(L, X). Player P wins the play < E_1, E_1, E_2, U_2.........> if {Un : n ∈ N} covers X, otherwise player E wins.

A finite sequence $<E_1, U_1$,........., En Un> of subsets in X is said to be admissible for G(L,

X) if the infinite sequence < E_1, U_1........En, Un, ¢, ¢..........> is a play of G(L, X).

A function s is said to be a strategy for player P in G (L, X) if the domain of s consists of the void sequence ¢ and the finite sequence <U_1,.........., Un> of open sets in X and if s (¢) and s (U_1, ..., Un) are closed in X and belong to L.

A strategy s for player P in the game G (L, X) is said to be winning if he wins each play <E_1, U_1, E_2, U_2,...........> in G (L, X) such that E_1 = s and En+1 = s (U_1, ..., Un), for all n ∈ N.

The Following notations are used:

DL – The class of all spaces which have a discrete closed cover consisting of members of L.

FL – The class of all spaces which have a finite closed cover consisting of members of L.

C – The class of all compact spaces.

Cm – The class of m-compact space.

I_1, I_2 – Arbitary classes of spaces possessing hereditary property s.t.

I_1 x I_2 = {X x Y : X ∈ I_1 and Y ∈ I_2}.

We define the following two products spaces:

Def 1 : D-Product: A Product space X x Y is said to be a D-product if for each closed set M of X x Y and each open set O of X x Y with M O, there is a o-discrete collection J by closed rectangles in X x Y such that M J O.

For a closed rectangle R in X x Y, R| and R|| denote the projection of R into X and Y respectively. Thus R is a closed rectangle in X x Y iff R' and R" are closed in X & Y and R is an open rectangle in X x Y iff R| and R|| are open in X and Y such that R = R| and R||.

Def 2: C-Product space X x Y is said to be a C-product if for each closed set M of X x Y and each open set O of X x Y with M O there is a countable collection J by closed rectangles in X x Y such that M J O.

The following result follows easily.Theorem: (1) Let X and Y be spaces such that X x Y is a D-Product. If player P has winning strategies in G (I_1, X) and G (I_2, Y), then he has a winning strategy in G (D (I_1 x I_2), X x Y).

Now, we prove the following.

Theorem: (2) Let X be a collection wise normal space and Y a subparcompact space with χ (Y) ≤ m. If player P has a winning strategy in G (DCm, X), then every open cover of X x Y with power ≤ m has a o-discrete refinement by closed rectangles in X x Y.

Proof: Let s be a winning strategy of player P in G (DCm, X). Let C be an arbitrary open cover of X x Y with ¦c¦ ≤ m.

We construct:

i. a sequence {Jn : n ≥ 0} collections of closed rectangles in X x Y;

ii. sequence {<ℜn, ≤ Ψn> : n ≥ 0} of the pairs of collections Rn by closed rectangles in X x Y;

iii. the function Ψn: ℜn?ℜn-1 ; satisfying the following five conditions:

a) Jn is o-discrete in X x Y.

b) ℜn is o-discrete in X x Y.

c) Each F ∈ Jn is contained in some G ∈ C.

d) If (x, y) ∈ Rn-1 ∈ ℜn-1 and (x, y) ¢ Jn.

Then there is Rn ∈ ℜn such that (x, y) ∈ Rn and Ψn (Rn) = Rn-1.

e) for an R ∈ ℜn, let Un = X – R and Uk = X – (Ψk+1 o.......o Ψn (R))', for 1 ≤ k ≤ n – 1.

We put E_1= s (ϕ) and E_{k+1} = s (U_1,.........,Uk) for 1 ≤ k ≤ n – 1.

Then the finite sequence < E_1, U_1 ... En, Un, > is admissible for G (DCm, X).

Let J_0 = {ϕ} and $ℜ_0$ = {X x Y}.We suppose that the above {Ji, : i ≤ n} and {< ℜi, Ψi > : i ≤ n } are already constructed.

We pick and R ∈ ℜn.

Let < E_1, U_1, ..., En, Un > be the admissible sequence in G (DCm, X).

Hence there is a discrete collection {Cα : α ∈

Ω R} by m-compact closed sets in R' such that s (U_1, ..., Un) ? R' = ∪ {Cα : α ∈ Ω R} . We can a choose discrete colletion {Wα : α ∈ Ω (R) } of open sets in R' s.t. Cα ⊂ Wα, for all α ∈ Ω (R).
Since Cα is m-compact, ¦c¦ ≤ m, χ(Y) ≤ m and R" is subparacompact.
There is a collection J^{α}_{n+1} = {CI $U^{\alpha, i}_{\lambda}$ x H_{λ} : i = I,......k_{λ} and λ ∈ ^ (k)} by closed rectangle in R. which satisfying the following four conditions:

(1) Each $U^{\alpha, i}_{\lambda}$ is open in R'.
(2) C_{α} ⊂ ∪ { $U^{\alpha, i}_{\lambda}$: i = 1, ...k_{λ}} ⊂ W_{α}
(3) Each C1 $U^{\alpha, i}_{\lambda}$ x H_{λ} is contained in some G ∈ C.
(4) {H_{λ} : ∈ ^(α)} is a σ-discrete closed cover of R". Then J_{n+1} (R) = ∪{ J^{α}_{n+1} : α∈Ω(R)} is a σ-discrete in X x Y.

Put $U^{\alpha,}_{\lambda}$ = {CIW_{α} - ∪ { $U^{\alpha, i}_{\lambda}$: i ≤ i ≤ k_{λ}} x H_{λ} }, for all $_{\lambda}$ ∈ ^ (k)
Again put R = (R'-∪ {Wα : α ϵ Ω ®} x R".
Moreover, we put ℜn_{+1} (R) = {R ∪ { R^{α}_{λ} : λ ∈ (α)} and λ ∈ Ω(R)}.
Then ℜn_{+1} (R) is also σ-discrete collection by closed rectangles in R.
We set Jn_{+1} = ∪ { Jn_{+1} (R) : R ∈ ℜn} and ℜn_{+1} = ∪ { ℜn_{+1} (R) : R ∈ ℜn}.
The function Ψ_{n+1} : ℜn_{+1} ? Rn defined as Ψ_{n+1} (ℜn_{+1} (R)) = {R}, for all R ∈ ℜ.
From (a), Jn+1 and ℜn+1 are o-discrete in X x Y..
The conditions (a) and (b) are satisfied.
By (3), then the condition © is also satisfied.
The conditions (d) and (e) are very clear.
Let J = {Jn : n ∈ N}.
We can easily show that J is a cover of X x Y. Therefore J is a o-discrete refinement of C by closed rectangles in X x Y.
With the consequences of the above theorem and assuming PCm to be the class of all product spaces with the first factor being m-compact, the following can be obtained easily:I. Let X be a collectionwise normal space and Y be a subparacompact space with χ (Y) ≤ m. If player P has a winning strategy in G (DCm, X), then X x Y is a D-product.

II. Let X be a paracompact space and Y be a subparacompact space.

III. Let X be a collectionwise noromal space and Y be a subparacomact space with χ (Y) ≤ m. If player P has a winning strategy in G (DCm, X), then he has a winning strategy in G (D(PCmm), X x Y).

Fuzzy set coalition

A game is determined by information, decisions and goals. But human notions (ideas) and decisions are fuzzy. For, a man with immense entropy functions may err, set right and understanding a little may increases his understanding in the pursuit of some knowledge. Therefore, in a game, perfect information, decisions & goals may not be feasible. We are therefore, led to the introduction of fuzzy games.
Let G = (N, v) be a nonfuzzy game of the set N = {1, 2, 3, ..., n} of n players in which v : s ? R is a real valued function (characteristic function) from a family of coalition S N to the set of real numbers R. Hence v(A) means the gain which a coalition. A can acquire only through the action of A, the coalition A can be specified by the characteristic function τA (i) = {1 if i ∈ A;/0 if i ¢ A.
A rate of participation τA (i) of a player i is defined by
τA (i) = 1, if a player i participates in A and
τA (i) = 0, if a player i does not participate in A.
Consequently, a coalition A is represented by
τA = (τA (1), τA (2),…….τA (n)).
A fuzzy coalition τ is defined as a coalition in which a player i can participate with a rate of participation τ1 ∈ [0, 1] instead of {0,1}. The characteristic function of a fuzzy game is a real valued function f : [0.1]n ? R which specifies a real number f (τ) for any fuzzy coalition τ.
This fuzzy game is denoted by FG = (N, f).
By obtaining this fuzzy game, we can have the corresponding results of previous section easily which may produce better results.

Conclusion

In a field of decision theory, game theory has a remarkable importance. To play a game over a Hausdorff topological space, a new approach has been presented in this paper. Human ideas, decisions and goals determine a game but these notions are fuzzy in nature. Hence in a game, perfect information, decisions and goals may not be feasible. A good attempt has been made to apply fuzzy set theory to obtain feasible solution for a proposed problem of real life which may be a very useful tool for the researcher working in the field of fuzzy systems.

References

Engelking, R.(1968). *Outline of general topology*. Amsterdam: North Holland Publishing Company.

Kumar, B.P. (1982). *Lattice and topological approach to game theory,* Doctoral dissertation, Bhagalpur University, Bhagalpur, Bihar.

Von, Neumann (1947). *Theory of games and economic behavior* (2nd ed). Prinston: Prinston University Press.

Kumar, B.P. & Kumar, Sanjay (2004). Covering property of spaces defined by games and its fuzzification. *Proceedings of Advances in Mathematics and its Application.* ISMAMS.