Energy of Sitnikov's restricted three body problem if the primaries are source of radiation and triaxial rigid bodies

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Abstract

In this paper the joint effect of source of radiation and triaxial rigid body has been studied. The energy of Sitnikov's restricted three body problem when primaries are sources of radiation and energy of Sitnikov's restricted problem of three bodies when primaries are triaxial rigid bodies have been studied to calculate the joint effect. Equation of motion of the third body of infinitesimal mass, if primaries are sources of radiation and triaxial rigid bodies, are calculated.

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1. Introduction

Sitnikov [1] problem can be integrated in case of restricted problem if primaries are taken to be equal masses. If primaries are triaxial rigid bodies then distance between them will be changed. Then third body can move along z axis. But if primaries are sources of radiation then the third body moves along the line perpendicular through the centre of mass of the primaries and is perpendicular to the plane of motion.

Thapa et al. [2] obtained the periodicity of Sitnikov's restricted three body problem when the primaries are sources of radiation. They found the equations of motion of the third body and total energy of the system. R.R. Thapa et al. [3] also obtained the equations of motion of the Sitnikov's restricted problem of three bodies when Primaries are triaxial rigid bodies. They got the equations of motion in dimensionless variables and Cartesian form as

\[
\ddot{x} - 2n\dot{y} = \Omega_x
\]

\[
\ddot{y} + 2n\dot{x} = \Omega_y
\]

\[
\ddot{z} = \Omega_z
\]
where \( \Omega = \frac{\mu}{2} \left( \frac{r_1}{r_1^3} + \frac{1}{r_2^3} \right) + \frac{1-\mu}{r_1} + \frac{\mu}{2r_1} + \frac{(1-\mu)(2\sigma_1 - \sigma_2)}{2r_1^3} 
+ \frac{\mu(2\sigma_1' - \sigma_2')}{2r_1^3} \left( 3(1-\mu)(\sigma_1 - \sigma_2) \right) 
+ \frac{3(1-\mu)(\sigma_1 - \sigma_2)}{2r_1^3} 
+ \frac{3\mu(\sigma_1' - \sigma_2')}{2r_1^3} \frac{3(1-\mu)(\sigma_1^2 - \sigma_2^2)}{2r_1^3} 
+ \frac{3(1-\mu)(\sigma_1^2 - \sigma_2^2)}{2r_1^3} \frac{3mu\sigma_1'z^2}{2r_1^3}

In present work we have proposed to get the energy of restricted three body problem if the primaries are source of radiation and triaxial rigid bodies.

2. Equation of motion

We will adopt the notation and terminology of szehely [4].

We have the equation of motion of the third body of infinitesimal mass is

\[
\ddot{z} = \frac{\partial \Omega}{\partial z} = -(1-p)z + \frac{3(2\sigma_1 - \sigma_2 + 2\sigma_1' - \sigma_2')z}{\left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} + \frac{15(\sigma_1 + \sigma_1')z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}}
\]

\[
\frac{d^2 z}{dt^2} = \frac{-(1-p)z}{\left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{3\alpha z}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} + \frac{15(\sigma_1 + \sigma_1')z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}}
\]

where \( \alpha = 2\sigma_1 - \sigma_2 + 2\sigma_1' - \sigma_2' \)

Equation (1) is non-linear equation. It can be integrated by linearizing it.

3. Energy of the third body

If we multiply (1) by \( \frac{2dz}{dt} \) we get,

\[
\frac{dz^2}{dt^2} = \frac{2(1-p)z}{\left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} \frac{dz}{dt} - \frac{3\alpha z}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} \frac{dz}{dt} + \frac{15(\sigma_1 + \sigma_1')z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} \frac{2dz}{dt}
\]

Or,

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{dz}{dt} \right)^2 = \frac{(1-p)z}{\left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} \frac{dz}{dt} - \frac{3\alpha z}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} \frac{dz}{dt} + \frac{15(\sigma_1 + \sigma_1')}{\left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} z \frac{dz}{dt}
\]
If we substitute \( u \frac{du}{dt} = z \frac{dz}{dt} \) and \( u^2 = z^2 + \frac{a^2}{4} \) we get,

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{dz^2}{dt} \right) = -\frac{(1-p)u}{u^2} \frac{du}{dt} - \frac{3\alpha}{4u^2} \frac{du}{dt} + \frac{15(\sigma_i + \sigma_i')}{4} \left[ \frac{1}{u^2} \frac{du}{dt} - \frac{1}{4} \frac{a^2}{u^6} \frac{du}{dt} \right]
\]

\[
= -\frac{(1-p)u}{u^2} \frac{3\alpha}{4u^4} \frac{du}{dt} + \frac{15}{4} (\sigma_i + \sigma_i') \frac{1}{u^4} \frac{du}{dt} - \frac{15}{4} (\sigma_i + \sigma_i') \frac{a^2}{4} \frac{1}{u^6} \frac{du}{dt}
\]

\[
= (1-p)u^{-1} \frac{du}{dt} + \frac{6}{4} (\sigma_i + \sigma_i') u^{3-3} + \frac{3(\sigma_i + \sigma_i') a^2}{16} u^{5-5} + c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 = (1-p)u^{-1} + \frac{\alpha}{4u^3} + \frac{5}{4} (\sigma_i + \sigma_i') u^{-3} + \frac{3(\sigma_i + \sigma_i') a^2}{16} u^{-5} + c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 = \frac{1}{u} + \frac{\alpha}{4u^4} - \frac{5}{4} (\sigma_i + \sigma_i') u^{-3} + \frac{3(\sigma_i + \sigma_i') a^2}{16} u^{-5} + c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 = \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{\alpha}{4(\sqrt{z^2 + \frac{a^2}{4}})^3} - \frac{5(\sigma_i + \sigma_i')}{4(\sqrt{z^2 + \frac{a^2}{4}})^3} + \frac{3(\sigma_i + \sigma_i') a^2}{16 (\sqrt{z^2 + \frac{a^2}{4}})^3} + c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 = \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{\alpha}{4(\sqrt{z^2 + \frac{a^2}{4}})^3} + \frac{5(\sigma_i + \sigma_i')}{4(\sqrt{z^2 + \frac{a^2}{4}})^3} - \frac{3(\sigma_i + \sigma_i') a^2}{16 (\sqrt{z^2 + \frac{a^2}{4}})^3} = c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 = \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{5\sigma_i + 5\sigma_i' - 2\sigma_i + 2\sigma_i' - 2\sigma_i' + \sigma_i'}{4(\sqrt{z^2 + \frac{a^2}{4}})^3} - \frac{3(\sigma_i + \sigma_i') a^2}{16 (\sqrt{z^2 + \frac{a^2}{4}})^3} = c
\]
\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 - \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{3(\sigma_1 + \sigma_2 + 3\sigma_1 \sigma_2 + \sigma_2^2)}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{3(\sigma_1 + \sigma_2^3)z^2}{16 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}}} = c
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{dz}{dt} \right)^2 - \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{p}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{3(\sigma_1 + \sigma_2^3)z^2}{16 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}}} = \text{constant}
\]

\[
E = \frac{1}{2} \left( \frac{dz}{dt} \right)^2 - \frac{(1-p)}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{p}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{3(\sigma_1 + \sigma_2^3)z^2}{16 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}}} = \text{constant.}
\]

Now from equation (1)

\[
\frac{dz}{dt} + \frac{(1-p)z}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{3az}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{15(\sigma_1 + \sigma_2^3)z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}} z^3} = 0
\]

\[
\Rightarrow \frac{dz}{dt} + \frac{(1-p)z}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{3az}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{15(\sigma_1 + \sigma_2^3)z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}}} z^3 = 0
\]

\[
\Rightarrow \frac{dz}{dt} + \frac{(1-p)z}{\sqrt{z^2 + \frac{a^2}{4}}} + \frac{3az}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{3}{2}}} - \frac{15(\sigma_1 + \sigma_2^3)z^3}{4 \left( z^2 + \frac{a^2}{4} \right)^{\frac{5}{2}}} z^3 = 0
\]

\[
\Rightarrow \frac{d^2z}{dt^2} + \frac{(1-p)8z}{a^3} \left( \frac{4z^2}{a^3} \right)^{\frac{3}{2}} + \frac{24az}{a^3} \left( \frac{4z^2}{a^3} \right)^{\frac{5}{2}} - \frac{15(\sigma_1 + \sigma_2^3)z^3}{a^3} \left( \frac{4z^2}{a^3} \right)^{\frac{7}{2}} = 0
\]

\[
\text{as } z < a \text{ so } \left( \frac{z}{a} \right)^{\gg 1}
\]

\[
\Rightarrow \frac{d^2z}{dt^2} + \frac{(1-p)8z}{a^3} \left( \frac{3}{2} \left( \frac{2z}{a} \right)^2 \right) + \frac{24az}{a^3} \left( \frac{5}{2} \left( \frac{2z}{a} \right)^2 \right) - \frac{15 \times 32 z^3 (\sigma_1 + \sigma_2^3) \left( \frac{7}{2} \left( \frac{2z}{a} \right)^2 \right)}{a^3} = 0
\]

Neglecting higher order term of \((z/a)\) above the third order, we get,
\[
\frac{d^2 z}{dt^2} + \frac{(1 - p)8z - 8z}{a^3} + \frac{12}{2} \left( \frac{z}{a} \right)^2 + \frac{24\alpha z - 24\alpha z}{a^5} - \frac{5 \times 4z^2}{2a^2} - \frac{480(\sigma_i + \sigma_i')}{a^3} z^3 \\
+ \frac{480(\sigma_i + \sigma_i')}{a^3} z^3 \frac{7}{2} \times \frac{4z^2}{a^7} = 0
\]

\[
\frac{d^2 z}{dt^2} + \left( \frac{(1 - p)8 + 24\alpha}{a^3} \right) - \left( \frac{48}{a^5} + \frac{240\alpha}{a^7} + \frac{480(\sigma_i + \sigma_i')}{a^7} \right) z^3 + ... = 0
\]

\[
\frac{d^2 z}{dt^2} + \frac{8(1 - p)}{a^5} \left( a^2 + 3\alpha \right) z - \frac{48}{a^7} \left( a^2 + 5\alpha + 10(\sigma_i + \sigma_i') \right) z^3 = 0
\]

(3)

\[
\frac{d^2 z}{dt} + \eta_0^2 z - \epsilon z^3 = 0
\]

(4)

where \( \eta_0^2 = \frac{8(1 - p)}{a^5} \left( a^2 + 3\alpha \right) = \frac{8q(a^2 + 3\alpha)}{a^5} \)

and \( \epsilon = \frac{48}{a^7} \left[ a^2 + 5\alpha + 10(\sigma_i + \sigma_i') \right] \)

The equations (3), (4) represent energy of Sitnikov's restricted three body problem if the primaries are source of radiation and triaxial rigid bodies.

4. Result and Discussion

The energy of third body depends on variable \( z \), radiation parameter \( p \), (parameter) \( \alpha = 2\sigma_1 - \sigma_2 + 2\sigma_1 - \sigma_2 \), distance between the primaries \( a \); where \( \sigma_1 = \frac{a_1^2 - a_3^2}{2R^2} \), \( \sigma_2 = \frac{a_2^2 - a_3^2}{2R^2}, \sigma_1^1 = \frac{b_1^2 - b_3^2}{2R^2}, \sigma_2^1 = \frac{b_2^2 - b_3^2}{2R^2} \); \( R \) is dimensional distance between the primaries and \( a_1, a_2, a_3; b_1, b_2, b_3 \) are lengths of semi-axes of two primaries parallel to axes respectively. The kinetic energy of the third body depends on variable \( z \) but potential energy depends on \( \eta_0 \) and \( \epsilon \).

where \( \eta_0^2 = \frac{8q(a^2 + 3\alpha)}{a^5} \) and \( \epsilon = \frac{48}{a^7} \left[ a^2 + 5\alpha + 10(\sigma_i + \sigma_i') \right] \).

Hence the third body will perform oscillatory motion.

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