On contact conformal curvature tensor in trans-Sasakian manifolds

Riddhi Jung Shah
Department of Mathematics, Janata Campus
Nepal Sanskrit University, Dang
E-mail: shahrjgeo@gmail.com
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Abstract
The purpose of this paper is to study some results on contact conformal curvature tensor in trans-Sasakian manifolds. Contact conformally flat trans-Sasakian manifold, \( \xi \) -contact conformally flat trans-Sasakian manifold and curvature conditions \( C_0(\xi, X).S = 0 \) and \( C_0(\xi, X).C_0 = 0 \) are studied with some interesting results. Finally, we study an example of 3-dimensional trans-Sasakian manifold.

Keywords: Contact conformal curvature tensor; Trans-Sasakian manifold; Hermitian manifolds.

1. Introduction

In 1978, Gray and Hervella [1] studied on the sixteen classes of almost Hermitian manifolds and their linear invariants. They considered unitary group \( U(n) \) on a certain space \( W \) and studied that the representation of \( U(n) \) on \( W \) has four irreducible components, \( W = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \). From these four components sixteen different invariants subspaces were obtained. Among four components \( W_3 \oplus W_4 \) corresponds to the class of Hermitian manifolds. Oubina [2] studied a new class of almost contact metric structure, called trans-Sasakian which is an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold. An almost contact metric structure \( (\phi, \xi, \eta, g) \) (where \( \phi \) is a \((1, 1)\) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form and \( g \) is a compatible Riemannian metric) on \( M \) is trans-Sasakian [2] if \((M \times \mathbb{R}, J, G)\) belongs to the class \( W_4 \), where \( J \) is the almost complex structure on \( M \times \mathbb{R} \) defined by

\[
J(X, f \frac{d}{dt}) = \left(\phi X - f \xi, \eta(X) \frac{d}{dt}\right),
\]
for any vector field $X$ on $M$, where $G$ is the product metric on $M \times \mathbb{R}$. Trans-Sasakian manifold is the trans-Sasakian structure of type $(\alpha, \beta)$, where $\alpha$ and $\beta$ are smooth functions on $M$. Trans-Sasakian manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are cosympletic [3], $\alpha$-Sasakian [4] and $\beta$-Kenmotsu manifold [4,5] respectively. Trans-Sasakian manifolds have been studied in [6,7] and by many others.

On the other hand, contact conformal curvature tensor field was introduced and defined by Jeong et al. [8] in a $(2n+1)$-dimensional Sasakian manifold which was constructed from the conformal curvature tensor field defined by Kitahara et al. [9] in a Kähler manifold by using the Boothby-Wang’s fibration. Contact conformal curvature tensor has also been studied in [10] and [11].

2. Preliminaries

Let $M$ be a $(2n+1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$-is a 1-form and $g$ is a compatible Riemannian metric such that [3]

(2.1) \[ \phi^2 (X) = -X + \eta(X) \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \]

(2.2) \[ g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y), \]

(2.3) \[ g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \]

for all $X, Y \in TM$. The fundamental 2-form $\Phi$ of the almost contact metric structure $(\phi, \xi, \eta, g)$ is defined as

(2.4) \[ \Phi(X, Y) = g(X, \phi Y) = -g(\phi X, Y), \]

since $\phi$ is a skew-symmetric with respect to $g$.

An almost contact metric manifold $M$ is called trans-Sasakian manifold if [2]

(2.5) \[ \{ \nabla_X \phi \} Y = \alpha \{ g(X, Y) \xi - \eta(Y) X \} + \beta \{ g(\phi X, Y) \xi - \eta(Y) \phi X \}, \]

where $\nabla$ is Levi-Civita connection of Riemannian metric $g$ and $\alpha, \beta$ are smooth functions on $M$. From (2.5) it follows that

(2.6) \[ \nabla_X \xi = -\alpha \phi X + \beta \{ X - \eta(X) \xi \}, \]

(2.7) \[ \nabla_X \eta = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \]

In a $(2n+1)$-dimensional trans-Sasakian manifold $M$, the following relations hold [6]

(2.8) \[ R(X, Y) \xi = (\alpha^2 - \beta^2)[\eta(Y) X - \eta(X) Y] - (X \alpha) \phi Y - (X \beta) \phi^2 (Y) \]

\[ + 2\alpha \beta \eta(Y) \phi X - \eta(X) \phi Y + (Y \alpha) \phi X + (Y \beta) \phi^2 (X) \],

(2.9) \[ R(\xi, X) Y = (\alpha^2 - \beta^2)[g(X, Y) \xi - \eta(Y) X] + 2\alpha \beta [g(\phi Y, X) \xi \]

\[ - \eta(Y) \phi X] + (Y \alpha) \phi X + g(\phi Y, X) (\text{grad} \alpha) \]

\[ + (Y \beta) [X - \eta(X) \xi] - g(\phi X, \phi Y) (\text{grad} \beta), \]
\begin{equation}
2\alpha \beta + (\xi \alpha) = 0, 
\end{equation}

\begin{equation}
S(X, \xi) = [2n(\alpha^2 - \beta^2) - (\xi \beta)]\eta(X) - ((\phi X) \alpha) - (2n-1)(X \beta),
\end{equation}

\begin{equation}
\eta(R(X, Y)Z) = -g(R(X, Y)\xi, Z),
\end{equation}

\begin{equation}
\eta(R(X, Y)\xi) = \eta(R(\xi, X)\xi) = \eta(R(\xi, \xi)\xi) = 0,
\end{equation}

\begin{equation}
\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2)(\xi \beta)]g(\phi X, \phi Y).
\end{equation}

In a \((2n + 1)\)-dimensional trans-Sasakian manifold if we put \(\phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta\), then we have

\begin{equation}
(\xi \beta) = 0,
\end{equation}

\begin{equation}
S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X),
\end{equation}

\begin{equation}
\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2)g(\phi X, \phi Y),
\end{equation}

\begin{equation}
R(\xi, X)\xi = (\alpha^2 - \beta^2)\{\eta(X)\xi - X\},
\end{equation}

\begin{equation}
R(X, \xi)\xi = -R(\xi, X)\xi.
\end{equation}

Throughout the paper we consider the trans-Sasakian manifold under the condition \(\phi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta\).

In a \((2n + 1)\)-dimensional trans-Sasakian manifold the contact conformal curvature tensor field \(C_0\) of type \((1, 3)\) which is defined by \([8]\) can be written as

\begin{equation}
C_0(X, Y)Z = R(X, Y)Z + \frac{1}{2n} \{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX
\end{equation}

\begin{equation}
- g(X, Z)QY + S(X, Z)\eta(Y)\xi - S(Y, Z)\eta(X)\xi
\end{equation}

\begin{equation}
+ \eta(X)\eta(Z)QY - \eta(Y)\eta(Z)QX + S(\phi X, Z)\phi Y
\end{equation}

\begin{equation}
- S(\phi Y, Z)\phi X + g(X, \phi Z)Q(\phi Y) - g(Y, \phi Z)Q(\phi X)
\end{equation}

\begin{equation}
+ 2g(X, \phi Y)Q(\phi Z) + 2S(\phi X, Y)\phi Z
\end{equation}

\begin{equation}
+ \frac{1}{2n(n+1)} \{2n^2 - n - 2 + \frac{(n+2)r}{2n}\} \{g(Y, \phi Z)\phi X
\end{equation}

\begin{equation}
- g(X, \phi Z)\phi Y - 2g(X, \phi Y)\phi Z
\end{equation}

\begin{equation}
+ \frac{1}{2n(n+1)} \{n+2 - \frac{(3n+2)r}{2n}\} \{g(Y, Z)X - g(X, Z)Y
\end{equation}

\begin{equation}
- \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\},
\end{equation}

where \(R, S, Q\) and \(r\) denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

From (2.20), we also have
\begin{align}
(2.21) & \quad C_0(X,Y)\xi = R(X,Y)\xi + (\alpha^2 - \beta^2 - 2)(\eta(Y)X - \eta(X)Y), \\
(2.22) & \quad C_0(\xi,X)Y = R(\xi,X)Y + (\alpha^2 - \beta^2 - 2)\{g(X,Y)\xi - \eta(Y)X\},
\end{align}
\begin{align}
\eta(C_0(X,Y)Z) &= \eta(R(X,Y)Z) \\
&\quad + (\alpha^2 - \beta^2 - 2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\},
\end{align}
(2.25) \quad \eta(C_0(\xi,X)Y) = 2(\alpha^2 - \beta^2 - 1)\{g(X,Y) - \eta(X)\eta(Y)\}.

**Definition.** A \((2n+1)\)-dimensional trans-Sasakian manifold \(M\) is said to be an \(\eta\)-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is of the form
\begin{equation}
S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),
\end{equation}
where \(a, b\) are smooth functions on \(M\). If \(b = 0\), then the manifold \(M\) becomes an Einstein manifold.

3. Contact Conformally Flat Trans-Sasakian Manifold

**Definition.** A \((2n+1)\)-dimensional trans-Sasakian manifold is said to be contact conformally flat if it satisfies the condition
\begin{equation}
C_0(X,Y)Z = 0.
\end{equation}

Now, we prove the following result:

**Theorem 3.1.** If a \((2n+1)\)-dimensional trans-Sasakian manifold \(M\) is contact conformally flat, then \(\alpha^2 = \beta^2 + 1\).

**Proof.** Let \(M\) be a \((2n+1)\)-dimensional trans-Sasakian manifold. Suppose \(M\) is contact conformally flat then the condition \(C_0(X,Y)Z = 0\) holds. Now, using (3.1) in (2.20) and taking inner product on both sides by \(\xi\), we get
\begin{align}
\eta(R(X,Y)Z) &= \frac{1}{2n}\{g(X,Z)S(Y,\xi) - g(Y,Z)S(X,\xi) \\
&\quad - \eta(X)\eta(Z)S(Y,\xi) + \eta(Y)\eta(Z)S(X,\xi)\} \\
&\quad + 2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}.
\end{align}

In view of (2.12), (2.16) and (3.2), we get
\begin{align}
0 &= 2(\alpha^2 - \beta^2 - 1)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + 2\alpha\beta[g(\phi Y,Z)\eta(Y) \\
&\quad - g(\phi X,Z)\eta(Y)] + (X\alpha)g(\phi Y,Z) - (Y\alpha)g(\phi X,Z) - (X\beta)g(\phi Y,\phi Z).
\end{align}

Putting \(X = \xi\) in (3.3) and using (2.1), (2.10) and (2.15), we obtain
\begin{equation}
(\alpha^2 - \beta^2 - 1)[g(Y,Z) - \eta(Y)\eta(Z)] = 0.
\end{equation}
Since \(g(Y,Z) - \eta(Y)\eta(Z) \neq 0\), we have \(\alpha^2 - \beta^2 - 1 = 0\). This implies that
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(3.5) \[ \alpha^2 = \beta^2 + 1. \]
This completes the proof of the theorem.

4. \(\xi\)-Contact Conformally Flat Trans-Sasakian Manifold

**Definition.** A trans-Sasakian manifold of dimension \((2n+1)\) is said to be \(\xi\)-contact conformally flat if the condition

\[ C_0(X, Y)\xi = 0 \]  

holds.

**Theorem 4.1.** Let \(M\) be a \((2n+1)\)-dimensional trans-Sasakian manifold satisfying the condition

\[ C_0(X, Y)\xi = 0, \]  

then \(\alpha^2 = \beta^2 + 1.\)

**Proof.** Let us consider a \((2n+1)\)-dimensional trans-Sasakian manifold \(M\) which satisfies the condition \(C_0(X, Y)\xi = 0.\) Then by virtue of (2.1), (2.3), (2.8), (2.16) and (4.1) in (2.20), we get

\[ \begin{align*}
0 &= 2(\alpha^2 - \beta^2 - 1)\{\eta(Y)X - \eta(X)Y\} - (X\alpha)\phi Y + (X\beta)Y \\
&\quad - (X\beta)\eta(Y)\xi + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X \\
&\quad - (Y\beta)X + (Y\beta)\eta(X)\xi.
\end{align*} \]  

Putting \(X = \xi\) in (4.2) and using (2.1), (2.10) and (2.15), we obtain

\[ \begin{align*}
(\alpha^2 - \beta^2 - 1)\{\eta(Y)\xi - Y\} &= 0.
\end{align*} \]

Since \(\eta(Y)\xi - Y = \phi^2(Y) \neq 0\), we have \((\alpha^2 - \beta^2 - 1) = 0.\) This yields

\[ \alpha^2 = \beta^2 + 1. \]
Thus the theorem is proved.

From theorem 3.1 and theorem 4.1, we can state the following result:

**Theorem 4.2.** Trans-Sasakian manifolds of dimension \((2n+1)\) which satisfy the conditions

\[ C_0(X, Y)Z = 0 \]  
and \(C_0(X, Y)\xi = 0\) are equivalent.

5. Trans-Sasakian Manifold Satisfying \(C_0(\xi, X).S = 0\)

Consider a trans-Sasakian manifold \(M\) of dimension \((2n+1)\). Let \(S\) be the Ricci tensor of type \((0, 2)\). We prove the following result:

**Theorem 5.1.** Let \(M\) be a \((2n+1)\)-dimensional trans-Sasakian manifold. If \(M\) satisfies the condition \(C_0(\xi, X).S = 0\), then it is an Einstein manifold with scalar curvature

\[ r = 2n(2n+1)(\alpha^2 - \beta^2); \]

**Proof.** Let \(M\) be a \((2n+1)\)-dimensional trans-Sasakian manifold which satisfies the condition
(5.1) \[ C_0(\xi, X)S(U, V) = 0. \]

This condition implies that

(5.2) \[ S(C_0(\xi, X)U, V) + S(U, C_0(\xi, X)V) = 0. \]

Putting \( V = \xi \) in (5.2) and using (2.16) and (2.22), we obtain

(5.3) \[ S(X, U) = 2n(\alpha^2 - \beta^2)g(X, U). \]

Taking an orthonormal frame field at any point of the manifold and contracting over \( X \) and \( U \) in (5.3), we get

(5.4) \[ r = 2n(2n+1)(\alpha^2 - \beta^2). \]

From (5.3) and (5.4) it follows that the manifold \( M \) is an Einstein manifold with scalar curvature \( r = 2n(2n+1)(\alpha^2 - \beta^2). \) This completes the proof of the result.

6. Trans-Sasakian Manifold Satisfying \( C_0(\xi, X)C_0 = 0 \)

Let \( M \) be a \( (2n+1) \)-dimensional trans-Sasakian manifold. Suppose the condition \( (C_0(\xi, X)C_0)(U, V)Z = 0 \) holds in \( M \). Then we have

Theorem 6.1. A \( (2n+1) \)-dimensional trans-Sasakian manifold satisfying the condition \( C_0(\xi, X)C_0 = 0 \) is contact conformally semi-symmetric if

\[ 0 = (\alpha^2 - \beta^2)\{2g(V, Z)X - g(X, V)Z - g(X, Z)V\} \]
\[ - g(R(\xi, V)Z, X)\xi - R(X, V)Z. \]

Proof. Let us consider a \( (2n+1) \)-dimensional trans-Sasakian manifold which satisfies the condition \( C_0(\xi, X)C_0(U, V)Z = 0 \), then by definition we have

(6.1) \[ 0 = C_0(\xi, X)C_0(U, V)Z - C_0(C_0(\xi, X)U, V)Z \]
\[ - C_0(U, C_0(\xi, X)V)Z - C_0(U, V)C_0(\xi, X)Z. \]

Using (2.22) in (6.1) we get

(6.2) \[ 0 = R(\xi, X)C_0(U, V)Z + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U, V)Z)\xi \]
\[ - \eta(C_0(U, V)Z)X - g(X, U)C_0(\xi, V)Z + \eta(U)C_0(X, V)Z \]
\[ - g(X, V)C_0(U, \xi)Z + \eta(V)C_0(U, X)Z - g(X, Z)C_0(U, V)\xi \]
\[ + \eta(Z)C_0(U, V)X]. \]

Taking inner product on both sides of (6.2) by \( \xi \) and using (2.24), we obtain
Putting $U = \xi$ in (6.3) and using (2.22), (2.23) and (2.25), we get
\begin{equation}
0 = g(R(\xi, X)C_0(U, V)Z, \xi) + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U, V)Z) - \eta(X)\eta(C_0(U, V)Z) - g(X, U)\eta(C_0(\xi, V)Z) + \eta(U)\eta(C_0(X, V)Z) - g(X, V)\eta(C_0(U, \xi)Z) + \eta(V)\eta(C_0(U, X)Z) - \eta(Z)\eta(C_0(U, V)X)].
\end{equation}

This implies that
\begin{equation}
R(\xi, X)C_0(\xi, V)Z = (\alpha^2 - \beta^2 - 2)[(\alpha^2 - \beta^2)2g(V, Z)X - g(X, V)Z - g(X, Z)V] - g(X, Z)V - g(R(\xi, V)Z, X)\xi - R(X, V)Z.
\end{equation}

From this it follows that the manifold is contact conformally semi-symmetric if the right hand side of (6.5) vanishes, i.e., if
\begin{equation}
0 = (\alpha^2 - \beta^2)(2g(V, Z)X - g(X, V)Z - g(X, Z)V)
- g(R(\xi, V)Z, X)\xi - R(X, V)Z.
\end{equation}

This completes the proof of the theorem.

**7. An Example of a 3-dimensional Trans-Sasakian Manifold**

Let us consider a 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields
\begin{equation}
e_1 = e^2\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right),
e_2 = e^2\frac{\partial}{\partial y},
e_3 = \frac{\partial}{\partial z},
\end{equation}

which are linearly independent at each point of $M$. Now we define a semi-Riemannian metric $g$ on $M$ as
\begin{equation}
ge(e_1, e_1) = g(e_2, e_1) = g(e_1, e_2) = 0,
ge(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\end{equation}

Let $\eta$ be a 1-form defined by $\eta(Z) = g(Z, e_3)$ for any vector field $Z \in M$ and $\varphi$ be a $(1, 1)$ tensor field defined by
(7.4) \( \varphi(e_1) = e_2, \varphi(e_2) = -e_1, \varphi(e_3) = 0. \)

The linearity property of \( \varphi \) and \( g \) yields that

(7.5) \( \eta(e_3) = 1, \varphi^2(Z) = -Z + \eta(Z)e_3, g(\varphi Z, \varphi U) = g(Z, U) - \eta(Z)\eta(U) \)

for any \( Z, U \in M \).

If we take \( e_3 = \xi \) in (7.5), \( (\varphi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \).

By the definition of Lie bracket and (7.1) we have

\[
[e_1, e_2] = e_1e_2 - e_2e_1 = e^z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right) e^z \frac{\partial}{\partial y} - e^z \frac{\partial}{\partial y} \left( e^z \frac{\partial}{\partial x} + ye^z \frac{\partial}{\partial z} \right)
= ye^z e_2 - e^{2z} e_3.
\]

Proceeding same way we obtain \([e_2, e_3] = -e_1 \) and \([e_1, e_3] = -e_2 \). Thus we have

(7.6) \[
[e_1, e_2] = ye^z e_2 - e^{2z} e_3, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1.
\]

Let \( \nabla \) be the Levi-Civita connection with respect to \( g \) then we have the Koszul's formula

(7.7) \[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)
+ g(\left[X, Y\right], Z) - g(\left[Y, Z\right], X) + g(\left[Z, X\right], Y).
\]

By the use of (7.2), (7.3) and (7.6), (7.7) yields

\[
\begin{align*}
\nabla_{e_1} e_3 &= -e_1 + \frac{1}{2} e^{2z} e_2, \quad \nabla_{e_2} e_2 = \frac{1}{2} e^{2z} e_3, \quad \nabla_{e_1} e_1 = e_3, \\
\nabla_{e_2} e_3 &= -e_2 - \frac{1}{2} e^{2z} e_1, \quad \nabla_{e_2} e_2 = e_3 + ye^z e_1, \quad \nabla_{e_2} e_1 = -ye^z e_2 + \frac{1}{2} e^{2z} e_1, \\
\nabla_{e_3} e_3 &= 0, \quad \nabla_{e_2} e_2 = -\frac{1}{2} e^{2z} e_1, \quad \nabla_{e_2} e_1 = \frac{1}{2} e^{2z} e_2.
\end{align*}
\]

In view of (2.6), (7.2), (7.3) and (7.4) we have

\[
\nabla_{e_1} \xi = \beta e_1 - \alpha e_2, \quad \nabla_{e_2} \xi = \alpha e_1 + \beta e_2 \quad \text{and} \quad \nabla_{e_3} \xi = 0 \quad \text{for} \quad e_3 = \xi.
\]

Comparing these equations with (7.8) (first column), we get \( \alpha = -\frac{1}{2} e^{2z} \) and \( \beta = -1 \).

Again, by virtue of (2.7) and \( \left( \nabla_X \eta \right) Y = \nabla_X \eta(Y) - \eta(\nabla_X Y) \) we obtain

\[
\left( \nabla_{e_1} \eta \right) e_1 = \beta = -1, \quad \left( \nabla_{e_2} \eta \right) e_1 = \alpha = -\frac{1}{2} e^{2z}, \quad \left( \nabla_{e_3} \eta \right) e_1 = 0.
\]

Thus from above calculation the conditions (2.6) and (2.7) are satisfied and the structure \( (\varphi, \xi, \eta, g) \)

is a trans-Sasakian structure of type \( (\alpha, \beta) \) where \( \alpha = -\frac{1}{2} e^{2z} \) and \( \beta = -1 \). Consequently

\( M^3(\varphi, \xi, \eta, g) \) is a trans-Sasakian manifold.
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