On the critical points of a polynomial

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Abstract
Let $P_n$ denote the set of all polynomials of the form

$$P(z) = (z - \alpha)\prod_{j=1}^{n-1}(z - z_j)$$

with $|\alpha| \leq 1$ and $|z_j| \geq 1$, $1 \leq j \leq n - 1$. In this paper, we show that $P'(z) \neq 0$ in $|z - \left(\frac{n-1}{n}\right)\alpha| < \frac{1}{n}$ for all polynomials $P \in P_n$. For $\alpha = 0$, this reduces to a result due to Aziz and Zargar.

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1. Introduction

The Gauss-Lucas Theorem states that if $S$ is the set of zeros of a polynomial

$$P(z) = \prod_{j=1}^{n}(z - z_j),$$

then every zero of the derivative $P'(z)$ is contained in the smallest convex set that contains $S$. This is best possible, in the sense that, if $P(z)$ has all its zeros in $D = \{z: |z| \leq 1\}$, then no proper subset of $D$ can be guaranteed to contain even one zero of $P'(z)$, (as is shown by the polynomial of the form $P(z) = (z - \alpha)^n$, since $P'(z) = n(z - \alpha)^{n-1}$ has zeros only at $\alpha$, which can lie anywhere in $D$). Gauss-Lucas theorem has been rather thoroughly investigated [8] and sharpened in several ways. However, there is one related question that deserves attention, namely given one specific zero $z_n$ of $P(z)$, what can be said about a neighborhood of $z_n$ that will always contain a zero of $P'(z)$.
The following conjecture was made by Bulgarian Mathematician B L Sendov in 1962 but became later known as Ilief’s conjecture (See [4, problem 4.5] or [6, p.795]).

**Conjecture 1.** Let $P(x)$ be a polynomial of degree $n$ having all its zeros in the unit disk $|x| \leq 1$. If $a$ is any one of these zeros, then $P'(x)$ has at least one zero in the disk $|x - a| \leq 1$. Since in 1962, when it is first became known, Conjecture 1 has been the subject of more than thirty articles. However, it was fully verified only for polynomials of degree $n \leq 8$ (see [13]). A variety of special cases have been dealt with over the years (See [2, 7, 11] for references), among which we mention that of a polynomial with at most five distinct zeros [5], as well as Miller’s qualitative result [10], according to which those zeros of $P(x)$ lying sufficiently close to the unit circle satisfy an even stronger condition that the one stated in Sendov’s conjecture (See also [12]).

Another Stronger conjecture than that of Ilief was made in 1969 by Goodman, Rahman and Ratti [3].

**Conjecture 2.** Let $P(x)$ be a polynomial of degree $n$ having all its zeros in the unit disk $|x| \leq 1$. If $a$ is any one of these zeros, then $P'(x)$ has at least one zero in the disk

$$|x - \frac{a}{2}| \leq 1 - \frac{|a|}{2}.$$ 

Conjecture 2 has been proved when $|a| = 1$ [3], but some counter examples have been devised for case $|a| < 1$ by M. J. Miller [10].

Recently Aziz and Zargar [2] have proved the following result.

**Theorem A.** If $P(x) = x^n \prod_{j=1}^{n-1} (x - z_j)$ is a polynomial of degree $n$ with $|z_j| \geq 1$, $j = 1,2,\ldots,n-1$, then $P'(x)$ does not vanish in $|x| < \frac{1}{n}$.

In this paper we establish a generalized form of above theorem. In fact we prove the following interesting result which extracts that portion of complex plane in which the above polynomial does not vanish.

**Theorem 1.** Let

$$P(x) = (x - a) \prod_{j=1}^{n-1} (x - z_j)$$

be a polynomial of degree $n$ with $|a| \leq 1$ and $|z_j| \geq 1$, $j = 1,2,\ldots,n-1$, then $P'(x)$ does not vanish in the disk
\[|z - \left(\frac{(n-1)}{n}\right)\alpha| < \frac{1}{n}.\]

The result is best possible as shown by the polynomial

\[P(z) = (z - \alpha)(z - e^{i\alpha})^{n-1},\]

where \(0 \leq \alpha < 2\pi\). Further taking \(\alpha = 0\) we get Theorem A. By using a similar argument, we can prove the following more general result.

**Theorem 2.** Let

\[P(z) = (z - \alpha)^{n-k} \prod_{j=1}^{n-k}(z - z_j)\]

be a polynomial of degree \(n\) with \(|\alpha| \leq 1\) and \(|z_j| \geq 1\), \(j = 1, 2, \ldots, n-k\) where \(1 \leq k \leq n-1\).

Then \(P'(z)\) has \(k-1\) fold zero at \(z = \alpha\) and remaining \(n-k\) zeros of \(P'(z)\) lie in the region

\[|z - \left(\frac{(n-k)}{n}\right)\alpha| \geq \frac{k}{n}.\]

The result is best possible as shown by the polynomial

\[P(z) = (z - \alpha)^k(z - e^{i\alpha})^{n-k-1},\]

where \(0 \leq \alpha < 2\pi\).

For the proof of this theorem we need the following lemma which is the coincidence theorem of Walsh [8, P.62] (see also [1]).

**Lemma.** Let \(G(z_1, z_2, \ldots, z_n)\) be a symmetric \(n\)-linear form of total degree \(n\) in \(z_1, z_2, \ldots, z_n\) and let \(C\) be a circular region containing the \(n\) points \(w_1, w_2, \ldots, w_n\), then there exists at least one point \(\alpha\) belonging to \(C\) such that

\[G(\alpha, \alpha, \ldots, \alpha) = G(w_1, w_2, \ldots, w_n).\]

**Proof of Theorem 2.** By hypothesis,

\[P(z) = (z - \alpha)^{n-k} \prod_{j=1}^{n-k}(z - z_j)\]

where \(|\alpha| \leq 1\) and \(|z_j| \geq 1\), \(j = 1, 2, \ldots, n-1\). Let \(T(z) = \prod_{j=1}^{n-k}(z - z_j)\), then \(T(z)\) is a polynomial of degree \(n-k\), having all its zeros in \(|z| \geq 1\) and we have

\[P(z) = (z - \alpha)^k T(z).\]

This implies

\[(1) \quad P'(z) = k(z - \alpha)^{k-1} T(z) + (z - \alpha)^k T'(z).\]
If now \( w \) is any zero of \( P'(z) \), then from (1), we get

\[
 k(w - \alpha)^{k-1}T'(w) + (w - \alpha)kT'(w) = P'(w) = 0.
\]

This is an equation which is linear and symmetric in the zeros of \( T(z) \); that is, in \( z_2, z_3, \ldots, z_{n-1}. \)

Hence an application of the above lemma with circular region \( C = \{ z : |z| \geq 1 \} \) shows that \( w \) will also satisfy the equation obtained by substituting into the equation (2)

\[
 T(z) = (z - \alpha)^{k-1},
\]

where \( \alpha \) is suitably chosen point in the circular region \( \{ z : |z| \geq 1 \} \). That is \( w \) satisfies the equation

or equivalently

\[
 k(w - \alpha)^{k-1}(w - \alpha)^{n-k} + (w - \alpha)^k(n - k)(w - \alpha)^{n-k-1} = 0.
\]

Thus \( w \) has the values \( w = \alpha \) or \( w = \left( \frac{n - k}{n} \right) \alpha + \frac{k \alpha}{n} \) where \( \alpha \) is suitably chosen point in \( \{ z : |z| \geq 1 \} \). If \( w = \alpha \), then using the fact that \( |z| \geq 1 \), it follows that

\[
 \left| w - \left( \frac{n - k}{n} \right) \alpha \right| = \left| \alpha - \left( \frac{n - k}{n} \right) \alpha \right|
\]

\[
 \geq |\alpha| - \left( \frac{n - k}{n} \right) |\alpha|
\]

\[
 \geq 1 - \left( \frac{n - k}{n} \right) |\alpha|
\]

\[
 \geq 1 - \left( \frac{n - k}{n} \right)
\]

\[
 = \frac{k}{n}.
\]

If

\[
 w = \left( \frac{n - k}{n} \right) \alpha + \frac{k \alpha}{n},
\]

then clearly

\[
 \left| w - \left( \frac{n - k}{n} \right) \alpha \right| = \frac{k \alpha}{n} \geq \frac{k}{n}.
\]

Thus in any case

\[
 \left| w - \left( \frac{n - k}{n} \right) \alpha \right| \geq \frac{k}{n}.
\]

Since \( w \) is an arbitrary zero of \( P'(z) \), it follows that every zero of \( P'(z) \) lie in the disk

\[
 \left| w - \left( \frac{n - k}{n} \right) \alpha \right| \geq \frac{k}{n}.
\]

This completes the proof of Theorem 2.

**Corollary.** If we take \( k = 1 \) in Theorem 2, we get Theorem 1.
References