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Application of the new approach of generalized (G'/G)-expansion method to find exact solutions of nonlinear PDEs in mathematical physics

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Abstract

The exact solutions of nonlinear evolution equations (NLEEs) play a crucial role to make known the internal mechanism of complex physical phenomena. In this article, we construct the traveling wave solutions of the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation by means of the new approach of generalized (G'/G)-expansion method. Abundant traveling wave solutions with arbitrary parameters are successfully obtained by this method and the wave solutions are expressed in terms of the hyperbolic, trigonometric, and rational functions. It is shown that the new approach of generalized (G'/G)-expansion method is a powerful and concise mathematical tool for solving nonlinear partial differential equations.

Keywords: New approach of generalized (G'/G)-expansion method, ZK-BBM equation; homogeneous balance, traveling wave solutions, solitary wave solutions, nonlinear evolution equation.

1. Introduction

Nonlinear phenomena arise in several aspects of physics as well as other natural and applied sciences. Essentially all the fundamental equations in physical sciences are nonlinear and, in general, such NLEEs are often very complicated to solve explicitly. The exact solutions of NLEEs play an important role in the study of nonlinear physical phenomena. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations still have drawn a lot of interest by diverse group of scientists. In the past three decades, there has been significant progress in the development of finding effective methods for obtaining exact solutions of NLEEs. These methods are the Exp-function method [1-3], the generalized Riccati equation [4], the Miura transformation [5], the Jacobi elliptic function expansion method [6, 7], the Hirota's bilinear method [8], the sine-cosine method [9], the tanh-function method [10], the extended tanh-function method [11-12], the homogeneous balance method [13], the modified Exp-function method [14], the (G'/G)-expansion method [15-22], the improved (G'/G)-expansion method [23], the modified simple equation method [24-28], the inverse scattering transform [29] and so on.

Recently, Naher and Abdullah [30] established a highly effective extension of the (G'/G) expansion method, called the new generalized (G'/G) expansion method to obtain exact traveling wave solutions of NLEEs. The aim of this article is to look for new study relating to the new generalized (G'/G) expansion method for solving the renowned ZK-BBM equation to make obvious the effectiveness and usefulness of the method.

The outline of this paper is organized as follows: In Section 2, we give the description of the new generalized (G'/G) expansion method. In Section 3, we apply this method to the ZK-BBM equation. In Section 4, Discussions are given. Conclusions are given in Section 5.

2. Description of the new generalized (G'/G)-expansion method

Let us consider a general nonlinear PDE in the form

$$\Phi(v, v_t, v_x, v_{xx}, v_{tt}, v_{tx}, \ldots), \tag{1}$$

where $v = v(x,t)$ is an unknown function, Φ is a polynomial in $v(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives.

Step 1: We combine the real variables x and t by a complex variable η

$$v(x,t) = v(\eta), \quad \eta = x \pm V\,t, \tag{2}$$

where V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $v = v(\eta)$:

$$\psi(v, v', v'', v''', \cdots), \tag{3}$$

where ψ is a polynomial of v and its derivatives and the superscripts indicate the ordinary derivatives with respect to η.

Step 2: According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$v(\eta) = \sum_{i=0}^{N} \alpha_i (d+M)^i + \sum_{i=1}^{N} \beta_i (d+M)^{-i}, \tag{4}$$

where either α_N or β_N may be zero, but could be zero simultaneously, α_i $(i = 0,1,2,\cdots,N)$ and β_i $(i = 1,2,\cdots,N)$ and d are arbitrary constants to be determined and $M(\eta)$ is

$$M(\eta) = (G'/G) \tag{5}$$

where $G = G(\eta)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$AGG'' - BGG' - EG^2 - C\,(G')^2 = 0, \tag{6}$$

where the prime stands for derivative with respect to η; A, B, C and E are real parameters.

Step 4: To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (3).

Step 5: Substitute Eq. (4) and Eq. (6) including Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we obtain polynomials in $(d+M)^N$ $(N=0,1,2,\cdots)$ and $(d+M)^{-N}$ $(N=0,1,2,\cdots)$. Subsequently, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_i $(i=0,1,2,\cdots,N)$ and β_i $(i=1,2,\cdots,N)$, d and V.

Step 6: Suppose that the value of the constants α_i $(i=0,1,2,\cdots,N)$, β_i $(i=1,2,\cdots,N)$, d and V can be found by solving the algebraic equations obtained in Step 5. Since the general solutions of Eq. (6) are known to us, inserting the values of α_i $(i=0,1,2,\cdots,N)$, β_i $(i=1,2,\cdots,N)$, d and V into Eq. (4), we obtain more general type and new exact traveling wave solutions of the nonlinear partial differential equation (1).

Step 7: Using the general solution of Eq. (6), we have the following solutions of Eq. (5):

Family 1: When $B\neq 0$, $\omega = A-C$ and $\Omega = B^2+4E(A-C)>0$,

$$M(\eta)=\left(\frac{G'}{G}\right)=\frac{B}{2\omega}+\frac{\sqrt{\Omega}}{2\omega}\,\frac{C_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+C_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{C_1\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+C_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)} \tag{7}$$

Family 2: When $B\neq 0$, $\omega = A-C$ and $\Omega = B^2+4E(A-C)<0$,

$$M(\eta)=\left(\frac{G'}{G}\right)=\frac{B}{2\omega}+\frac{\sqrt{-\Omega}}{2\omega}\,\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)} \tag{8}$$

Family 3: When $B\neq 0$, $\omega = A-C$ and $\Omega = B^2+4E(A-C)=0$,

$$M(\eta)=\left(\frac{G'}{G}\right)=\frac{B}{2\omega}+\frac{C_2}{C_1+C_2\eta} \tag{9}$$

Family 4: When $B=0$, $\omega = A-C$ and $\Delta = \omega E>0$,

$$M(\eta)=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{\omega}\,\frac{C_1\sinh(\frac{\sqrt{\Delta}}{A}\eta)+C_2\cosh(\frac{\sqrt{\Delta}}{A}\eta)}{C_1\cosh(\frac{\sqrt{\Delta}}{A}\eta)+C_2\sinh(\frac{\sqrt{\Delta}}{A}\eta)} \tag{10}$$

Family 5: When $B=0$, $\omega = A-C$ and $\Delta = \omega E<0$,

$$M(\eta)=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{\omega}\,\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{A}\eta)+C_2\cos(\frac{\sqrt{-\Delta}}{A}\eta)}{C_1\cos(\frac{\sqrt{-\Delta}}{A}\eta)+C_2\sin(\frac{\sqrt{-\Delta}}{A}\eta)} \tag{11}$$

3. Application of the method

In this section, we will put forth the new generalized (G'/G) expansion method to construct many new and more general traveling wave solutions of the ZK-BBM equation. Let us consider the ZK-BBM equation,

$$v_t + v_x - 2\,a\,v\,v_x - b v_{xxt} = 0\,. \tag{12}$$

We will use the traveling wave transformation Eq. (2) into the Eq. (12), which yields:

$$(1+V)\,v' - 2\,a\,v\,v' - b\;V\,v''' = 0\,. \tag{13}$$

Integrating Eq. (13) with respect to η once yields

$$(1+V)\,v - a\,v^2 - b V\,v'' + K = 0, \tag{14}$$

where K is an integration constant which is to be determined.

Taking the homogeneous balance between highest order nonlinear term v^2 and linear term of the highest order v'' in Eq. (14), we obtain $N = 2$. Therefore, the solution of Eq. (14) is of the form:

$$v(\eta) = \alpha_0 + \alpha_1 (d+M) + \alpha_2 (d+M)^2 + \beta_1 (d+M)^{-1} + \beta_2 (d+M)^{-2}, \tag{15}$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2$ and d are constants to be determined.

Substituting Eq. (15) together with Eqs. (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d+M)^N$ $(N = 0,1,2,.......)$ and $(d+M)^{-N}$ $(N = 1,\ 2, \cdots)$. We collect each coefficient of these resulted polynomials to zero yields a set of simultaneous algebraic equations (for simplicity, the equations are not presented here) for α_0, α_1, α_2, β_1, β_2 d, K and V. Solving these algebraic equations with the help of computer algebra, we obtain following:

Set 1: $K = -\dfrac{1}{4aA^4}(-16b^2V^2E^2\omega^2 - 8b^2V^2B^2E\omega + A^4 - b^2V^2B^4 + V^2A^4 + 2VA^4)$,

$$\alpha_0 = -\frac{1}{2aA^2}(12bVd^2\omega^2 + 12bVBd\omega - 8bVE\omega - A^2 - VA^2 + bVB^2),\ V = V, \tag{16}$$

$\beta_1 = \dfrac{6bV}{aA^2}(2d^3\omega^2 + 3Bd^2\omega - 2Ed\omega + B^2 d - EB),\ \ d = d,\ \ \alpha_1 = 0,\ \ \alpha_2 = 0,$

$\beta_2 = -\dfrac{6bV}{aA^2}(d^4\omega^2 + 2bd^3\omega - 2Ed^2\omega - 2BdE + B^2d^2 + E^2)\,.$

where $\omega = A - C,\ V,\ d,\ A,\ B,\ C,\ E$ are free parameters.

Set 2: $K = -\dfrac{1}{4aA^4}(-16b^2V^2E^2\omega^2 - 8b^2V^2B^2E\omega + A^4 - b^2V^2B^4 + V^2A^4 + 2VA^4)$,

$$\alpha_0 = -\frac{1}{2aA^2}(12bVd^2\omega^2 + 12bVBd\omega - 8bVE\omega - A^2 - VA^2 + bVB^2),\ V = V, \tag{17}$$

$\alpha_1 = \dfrac{6bV}{aA^2}(2d\omega^2 + B\omega),\ \alpha_2 = -\dfrac{6bV\omega^2}{aA^2},\ \beta_1 = 0,\ \beta_2 = 0\,.$

where $\omega = A - C,\ V,\ d,\ A,\ B,\ C,\ E$ are free parameters.

Set 3: $K = -\dfrac{1}{4aA^4}((V+1)^2A^4 - 256b^2V^2A^2E^2\omega^2 - 128b^2V^2B^2E\omega - 16b^2V^2B^4)$,

$\alpha_0 = \dfrac{1}{2aA^2}((V+1)A^2 + 8bVE\omega + 2bVB^2),\ \alpha_2 = -\dfrac{6bV\omega^2}{aA^2},\ \alpha_1 = 0,\ V = V,$

$$\beta_2 = -\frac{3bV}{8aA^2\omega^2}(16E^2\omega^2 + 8EB^2\omega + B^4),\ \beta_1 = 0,\ d = -\frac{B}{2\omega} \tag{18}$$

where $\omega = A - C, V,\ A,\ B,\ C,\ E$ are free parameters.

For set 1, substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying, yields following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_1}(\eta) = \alpha_0 + \beta_1(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\coth(\frac{\sqrt{\Omega}}{2A}\eta))^{-1} + \beta_2(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\coth(\frac{\sqrt{\Omega}}{2A}\eta))^{-2}.$$

$$v_{1_2}(\eta) = \alpha_0 + \beta_1(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\tanh(\frac{\sqrt{\Omega}}{2A}\eta))^{-1} + \beta_2(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\tanh(\frac{\sqrt{\Omega}}{2A}\eta))^{-2}.$$

Substituting Eq. (16) into Eq. (15), along with Eq. (8) and simplifying, our exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_3}(\eta) = \alpha_0 + \beta_1(d + \frac{B}{2\omega} + \frac{\sqrt{-\Omega}}{2\omega}\cot(\frac{\sqrt{-\Omega}}{2A}\eta))^{-1} + \beta_2(d + \frac{B}{2\omega} + \frac{\sqrt{-\Omega}}{2\omega}\cot(\frac{\sqrt{-\Omega}}{2A}\eta))^{-2}.$$

$$v_{1_4}(\eta) = \alpha_0 + \beta_1(d + \frac{B}{2\omega} - \frac{\sqrt{-\Omega}}{2\omega}\tan(\frac{\sqrt{-\Omega}}{2A}\eta))^{-1} + \beta_2(d + \frac{B}{2\omega} - \frac{\sqrt{-\Omega}}{2\omega}\tan(\frac{\sqrt{-\Omega}}{2A}\eta))^{-2}.$$

Substituting Eq. (16) into Eq. (15), together with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{1_5}(\eta) = \alpha_0 + \beta_1(d + \frac{B}{2\omega} + \frac{C_2}{C_1 + C_2\eta})^{-1} + \beta_2(d + \frac{B}{2\omega} + \frac{C_2}{C_1 + C_2\eta})^{-2}.$$

Substituting Eq. (16) into Eq. (15), along with Eq. (10) and simplifying, we obtain following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_6}(\eta) = \alpha_0 + \beta_1(d + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta))^{-1} + \beta_2(d + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta))^{-2}.$$

$$v_{1_7}(\eta) = \alpha_0 + \beta_1(d + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta))^{-1} + \beta_2(d + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta))^{-2}.$$

Substituting Eq. (16) into Eq. (15), together with Eq. (11) and simplifying, our obtained exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_8}(\eta) = \alpha_0 + \beta_1(d + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}\eta))^{-1} + \beta_2(d + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}\eta))^{-2}$$

$$v_{1_9}(\eta) = \alpha_0 + \beta_1(d - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta))^{-1} + \beta_2(d - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta))^{-2},$$

where $\eta = x - Vt$.

Again for set 2, substituting Eq. (17) into Eq. (15), along with Eq. (7) and simplifying, our traveling wave solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{2_1}(\eta)=\frac{1}{2aA^2}(2bV(B^2+4E\omega)+A^2(V+1)-3bV\Omega\coth^2(\frac{\sqrt{\Omega}}{2A}\eta)),$$

$$v_{2_2}(\eta)=\frac{1}{2aA^2}(2bV(B^2+4E\omega)+A^2(V+1)-3bV\Omega\tanh^2(\frac{\sqrt{\Omega}}{2A}\eta)),$$

Substituting Eq. (17) into Eq. (15), along with Eq. (8) and simplifying yields exact solutions, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{2_3}(\eta)=\frac{1}{2aA^2}(2bV(B^2+4E\omega)+A^2(V+1)+3bV\Omega\cot^2(\frac{\sqrt{-\Omega}}{2A}\eta)),$$

$$v_{2_4}(\eta)=\frac{1}{2aA^2}(2bV(B^2+4E\omega)+A^2(V+1)+3bV\Omega\tan^2(\frac{\sqrt{-\Omega}}{2A}\eta)).$$

Substituting Eq. (17) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{2_5}(\eta)=\frac{1}{2aA^2}(2bV(B^2+4E\omega)+A^2(V+1)-12bV\omega^2(\frac{C_2}{C_1+C_2\eta})^2),$$

Substituting Eq. (17) into Eq. (15), together with Eq. (10) and simplifying, yields following traveling wave solutions, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{2_6}(\eta)=\frac{1}{2aA^2}(bV(-B^2+8E\omega)+A^2(V+1)+12bV\sqrt{\Delta}(B\coth(\frac{\sqrt{\Delta}}{A}\eta)-\sqrt{\Delta}\coth^2(\frac{\sqrt{\Delta}}{A}\eta))).$$

$$v_{2_7}(\eta)=\frac{1}{2aA^2}(bV(-B^2+8E\omega)+A^2(V+1)+12bV\sqrt{\Delta}(B\tanh(\frac{\sqrt{\Delta}}{A}\eta)-\sqrt{\Delta}\tanh^2(\frac{\sqrt{\Delta}}{A}\eta))).$$

Substituting Eq. (17) into Eq. (15), along with Eq. (11) and simplifying, our exact solutions become, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{2_8}(\eta)=\frac{1}{2aA^2}(bV(-B^2+8E\omega)+A^2(V+1)+12bV\sqrt{\Delta}(iB\cot(\frac{\sqrt{-\Delta}}{A}\eta)+\sqrt{\Delta}\cot^2(\frac{\sqrt{-\Delta}}{A}\eta))).$$

$$v_{2_9}(\eta)=\frac{1}{2aA^2}(bV(-B^2+8E\omega)+A^2(V+1)-12bV\sqrt{\Delta}(iB\tan(\frac{\sqrt{-\Delta}}{A}\eta)-\sqrt{\Delta}\tan^2(\frac{\sqrt{-\Delta}}{A}\eta)),$$

where $\eta=x-Vt$.

Similarly, for set 3, substituting Eq. (18) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{3_1}(\eta)=\alpha_0-\frac{3bV\Omega}{2aA^2}\coth^2(\frac{\sqrt{\Omega}}{2A}\eta)+\frac{4b_2\omega^2}{\Omega}\tanh^2(\frac{\sqrt{\Omega}}{2A}\eta).$$

$$v_{3_2}(\eta)=\alpha_0-\frac{3bV\Omega}{2aA^2}\tanh^2(\frac{\sqrt{\Omega}}{2A}\eta)+\frac{4b_2\omega^2}{\Omega}\coth^2(\frac{\sqrt{\Omega}}{2A}\eta).$$

Substituting Eq. (18) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{33}(\eta) = \alpha_0 + \frac{3bV\Omega}{2aA^2}\cot^2(\frac{\sqrt{-\Omega}}{2A}\eta) - \frac{4b_2\omega^2}{\Omega}\tan^2(\frac{\sqrt{-\Omega}}{2A}\eta).$$

$$v_{34}(\xi) = \alpha_0 + \frac{3bV\Omega}{2aA^2}\tan^2(\frac{\sqrt{-\Omega}}{2A}\eta) - \frac{4b_2\omega^2}{\Omega}\cot^2(\frac{\sqrt{-\Omega}}{2A}\eta).$$

Substituting Eq. (18) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{35}(\eta) = \alpha_0 - \frac{6bV\omega^2}{aA^2}(\frac{C_2}{C_1 + C_2\eta})^2 + \beta_2(\frac{C_2}{C_1 + C_2\eta})^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{36}(\eta) = \alpha_0 - \frac{6bV\omega^2}{aA^2}(\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta))^2 + \beta_2(\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta))^{-2}.$$

$$v_{37}(\eta) = a_0 - \frac{6bV\omega^2}{aA^2}(\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta))^2 + \beta_2(\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta))^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{38}(\eta) = \alpha_0 - \frac{6bV\omega^2}{aA^2}(\frac{-B}{2\omega} + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}\eta))^2 + \beta_2(\frac{-B}{2\omega} + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}\eta))^{-2}.$$

$$v_{39}(\eta) = \alpha_0 - \frac{6bV\omega^2}{aA^2}(\frac{-B}{2\omega} - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta))^2 + \beta_2(\frac{-B}{2\omega} - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta))^{-2}.$$

where $\eta = x - Vt$.

4. Discussions

The advantages and validity of the method over the basic (G'/G)-expansion method and the generalized and improved (G'/G)-expansion method have been discussed in the following.

Advantages: The crucial advantage of the new approach against the basic (G'/G)-expansion method and the generalized and improved (G'/G)-expansion method is that the method provides more general and abundant new exact traveling wave solutions with many real parameters. The exact solutions have its great importance to expose the inner mechanism of the complex physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the correctness of their results and help them in the stability analysis.

Validity: In Ref. [31] Zhang et al. used the linear ordinary differential equation (LODE) as auxiliary equation and traveling wave solutions presented in the form $u(\xi)=\sum_{i=-m}^{m} a_i (G'/G)^i$, where $a_m \neq 0$. Akbar et al. [17] used same auxiliary equation (LODE) and the solution is presented in the form $u(\xi)=\sum_{n=-m}^{m} \frac{e_{-n}}{(d+(G'/G))^n}$, where either e_{-m} or e_m may be zero, but both e_{-m} or e_m cannot be zero together. It is noteworthy to point out that some of our solutions are coincided with previously published results, if the parameters are taken particular values which validate our solutions. Moreover, In Ref. [31] Zhang et al. investigated the ZK-BBM equation to obtain exact solutions via the improved (G'/G)-expansion method and achieved only six solutions (A.1)-(A.6) (**see appendix A**). On the other hand, by using the generalized and improved (G'/G)-expansion method, Akbar et al. [17] obtained nine solutions (B.1)-(B.9) (**see appendix B**) of the ZK-BBM equation. But in this article twenty seven exact solutions including (A.1)-(A.6) and (B.1)-(B.2) of the ZK-BBM equation are constructed by applying the new approach of generalized (G'/G)-expansion method.

5. Conclusion

In this study, we considered the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation. We employed the new approach of generalized (G'/G)-expansion method for the exact solution to this equation and constructed some new solutions which are not found in the previous literature. The method offers solutions with free parameters that might be imperative to explain some intricate physical phenomena. This study shows that the new generalized (G'/G)-expansion method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. Also, we observe that the new generalized (G'/G)-expansion method is straightforward and can be applied to many other nonlinear evolution equations.

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Appendix A: Zhang et al. solutions [31]

Zhang et al. [31] established exact solutions of the well-known the ZK-BBM equation by using the improved (G'/G)-expansion method which are as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{1_1} = \frac{-6bV\mu^2}{a\left(-\left(\frac{\lambda}{2}\right) + \frac{1}{2}\left(\sqrt{\lambda^2-4\mu}\right)\left(\frac{C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\right)^2} - \frac{-6bV\mu\lambda}{a\left(-\left(\frac{\lambda}{2}\right) + \frac{1}{2}\left(\sqrt{\lambda^2-4\mu}\right)\left(\frac{C_2 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_1 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_2 \cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\right)} - \frac{bV\lambda^2 + 8bV\mu - V - 1}{2a}, \quad (A.1)$$

where C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_{1_2} = \frac{-6bV\mu^2}{a\left(-\left(\frac{\lambda}{2}\right) + \frac{1}{2}\left(\sqrt{4\mu-\lambda^2}\right)\left(\frac{-C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\right)^2} - \frac{-6bV\mu\lambda}{a\left(-\left(\frac{\lambda}{2}\right) + \frac{1}{2}\left(\sqrt{4\mu-\lambda^2}\right)\left(\frac{-C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\right)} - \frac{bV\lambda^2 + 8bV\mu - V - 1}{2a}, \quad (A.2)$$

where C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_{12} = \frac{-6bV\mu^2}{a(-(\frac{\lambda}{2}) + (\frac{C_2}{C_1 + C_2\xi}))^2} - \frac{-6bV\mu\lambda}{a(-(\frac{\lambda}{2}) + a(-(\frac{\lambda}{2}) + (\frac{C_2}{C_1 + C_2\xi}))} - \frac{bV\lambda^2 + 8bV\mu - V - 1}{2a}, \tag{A.3}$$

where C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu > 0$,

$$u_{21} = \frac{-3bV(\lambda^2 - 4\mu)}{2a}\left(\frac{C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 - \frac{bV\lambda^2 + 8bV\mu - V - 1}{2a}, \tag{A.4}$$

where C_1, C_2 are arbitrary constants.

For $C_1 > 0,\ C_1{}^2 < C_2{}^2$ above, then the solutions (A.4) turns into

$$u_{21} = \frac{-3bV(\lambda^2 - 4\mu)}{2a}\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2} + \xi_0\right) - \frac{bV\lambda^2 - 4bV\mu - V - 1}{2a}, \quad \xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2}\right),$$

When $\lambda^2 - 4\mu < 0$,

$$u_{22} = \frac{3bV(\lambda^2 - 4\mu)}{2a}\left(\frac{-C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2 + \frac{2bV\lambda^2 - 8bV\mu + V + 1}{2a}, \tag{A.5}$$

where C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_{22} = \frac{6bVC_2{}^2}{a(C_1 + C_2\xi)^2} + \frac{2bV\lambda^2 - 8bV\mu + V + 1}{2a}, \tag{A.6}$$

where C_1, C_2 are arbitrary constants.

Appendix B: Akbar et al. [17]

Akbar et al. [17] established a generalized and improved (G'/G)-expansion method and studied ZK-BBM equation, and obtained following solutions:

When $\lambda^2 - 4\mu > 0$,

$$u_{1_1}=\frac{-6bV\{\mu^2+d(\lambda^2 d+d^3-2\lambda d^2+2d\mu-2\lambda\mu)\}}{a(d-\frac{\lambda}{2}+\frac{1}{2}(\sqrt{\lambda^2-4\mu})(\frac{B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}))^2}-\frac{-6bV\{\mu\lambda-d(\lambda^2+2d^2-2\lambda d+2\mu)\}}{a(d-\frac{\lambda}{2}+\frac{1}{2}(\sqrt{\lambda^2-4\mu})(\frac{B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}))}-\frac{\{(bV\lambda^2+8bV\mu-V-1)+12bVd(d-\lambda)}{2a}, \quad (B.1)$$

where A, B are arbitrary constants.

When $\lambda^2-4\mu<0$,

$$u_{1_2}=\frac{-6bV\{\mu^2+d(\lambda^2 d+d^3-2\lambda d^2+2d\mu-2\lambda\mu)\}}{a(d-\frac{\lambda}{2}+\frac{1}{2}(\sqrt{4\mu-\lambda^2})(\frac{-B\sinh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A\cosh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{B\cosh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A\sinh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}))^2}-\frac{6bV\{\mu\lambda-d(\lambda^2+2d^2-2\lambda d+2\mu)\}}{a(d-\frac{\lambda}{2}+\frac{1}{2}(\sqrt{4\mu-\lambda^2})(\frac{-B\sinh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A\cosh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{B\cosh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A\sinh\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}))}-\frac{\{(bV\lambda^2+8bV\mu-V-1)+12bVd(d-\lambda)}{2a}, \quad (B.2)$$

where A, B are arbitrary constants.

When $\lambda^2-4\mu=0$,

$$u_{1_3}=\frac{-6bV\{\mu^2+d(\lambda^2 d+d^3-2\lambda d^2+2d\mu-2\lambda\mu)\}}{a(d-\frac{\lambda}{2}+(\frac{B}{A+B\xi}))^2}-\frac{\{(bV\lambda^2+8bV\mu-V-1)+12bVd(d-\lambda)}{2a}, \quad (B.3)$$

where A, B are arbitrary constants.

When $\lambda^2-4\mu>0$,

$$u_{2_1}=-\frac{3bV(\lambda^2-4\mu)}{2a}(\frac{B\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+A\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{B\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+A\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)})^2-\frac{2bV\lambda^2-8bV\mu+V+1}{2a}, \quad (B.4)$$

where A, B are arbitrary constants.

If $B > 0,\ A^2 < B^2$ above, then the solutions (B.4) turns into

$$u_{21} = \frac{3bV(\lambda^2 - 4\mu)}{2a} \sec h^2 \left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2} + \xi_0\right) - \frac{bV\lambda^2 - 4bV\mu - V - 1}{2a}, \quad \xi_0 = \tanh^{-1}\left(\frac{A}{B}\right),$$

When $\lambda^2 - 4\mu < 0$,

$$u_{2_2} = -\frac{3bV(4\mu - \lambda^2)}{2a}\left(\frac{-B\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + A\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{B\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + A\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right)^2 + \frac{2bV\lambda^2 - 8bV\mu + V + 1}{2a}, \qquad \text{(B.5)}$$

where A, B are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_{2_2} = -\frac{6bVB^2}{a(A + B\xi)^2} + \frac{2bV\lambda^2 - 8bV\mu + V + 1}{2a}, \qquad \text{(B.6)}$$

where A, B are arbitrary constants.

When $\lambda^2 - 4\mu > 0$,

$$u_{3_1} = \frac{-3bV(\lambda^2 - 4\mu)^2}{8a\left(d - \frac{\lambda}{2} + \frac{1}{2}(\sqrt{\lambda^2 - 4\mu})\left(\frac{B\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{B\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)\right)^2}$$

$$-\frac{6bV}{a}\left(d - \frac{\lambda}{2} + \frac{1}{2}(\sqrt{\lambda^2 - 4\mu})\left(\frac{B\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{B\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)\right)^2 \qquad \text{(B.7)}$$

$$-\frac{(2bV\lambda^2 - 8bV\mu V + V + 1)}{2a},$$

where A, B are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_{3_2} = -\frac{3bV(\lambda^2 - 4\mu)^2}{8a\left(d - \frac{\lambda}{2} + \frac{1}{2}(\sqrt{4\mu - \lambda^2})\left(\frac{-B\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + A\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{B\cos\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi) + A\sin\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi)}\right)\right)^2}$$

$$-\frac{6bV}{a}\left(d - \frac{\lambda}{2} + \frac{1}{2}(\sqrt{4\mu - \lambda^2})\left(\frac{-B\sin\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi) + A\cos\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi)}{B\cos\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi) + A\sin\frac{1}{2}(\sqrt{4\mu - \lambda^2}\xi)}\right)\right)^2 \qquad \text{(B.8)}$$

$$-\frac{(2bV\lambda^2 - 8bV\mu V + V + 1)}{2a},$$

where A, B are arbitrary constants.

When $\lambda^2 - 4\mu = 0,$

$$u_{33} = \frac{-3bV(\lambda^2 - 4\mu)^2}{8a(d - \frac{\lambda}{2} + (\frac{B}{A + B\xi}))^2} - \frac{6bV}{a}(d - \frac{\lambda}{2} + (\frac{B}{A + B\xi}))^2 + \frac{(V + 1 + 2bV\lambda^2 - 8bV\mu V)}{2a}, \tag{B.9}$$

where A, B are arbitrary constants.