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Bernoulli numbers and Euler's contributions to infinite series

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Abstract

This expository study focuses on the origin and mathematical development of the sequence of rational Bernoulli numbers. Johann Faulhaber developed foundational formulas for power sums of positive integers, which later enabled Leonhard Euler to systematically develop and apply Bernoulli numbers. This study presents Euler's use of Bernoulli numbers as a bridge between finite sums and infinite series, emphasizing their role in generating functions and in the evaluation of the Riemann zeta function at positive even integers. It further elucidates the relation between the zeta function and Bernoulli numbers and interprets how Leonhard Euler continued to expand the theory by deriving exponential generating functions, highlighting their significance in analytic number theory.

Keywords
Generating function, infinite series, Jacob Bernoulli, polynomials, Zeta function.

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1 Introduction

Johann Faulhaber (1580–1635) of Germany studied the sum of powers of positive integers in the finite case [1]. Following the formula developed by other previous mathematicians, he developed the formulae of finding the sum of series like $1^k + 2^k + 3^k + 4^k + \cdots + (n-1)^k$ up to the 17^{th} power which was published in his *Academia Algebrae* [2, 3]. But he couldn't generalize these particular formulae. Then this work was continued by the Swiss Mathematician Jacob Bernoulli (1654–1705). He successfully generalized Faulhaber's formulas for sums of powers of positive integers [3, 4].

Bernoulli became the first mathematician to realize the existence of a sequence of rational numbers $B_0, B_1, B_2, B_3, B_4, ...$ associated with the coefficients of terms of polynomials of sums of powers of integers. These numbers are very useful to establish a uniform formula for all sums of powers. Seki Takakazu (1642–1708) also developed the same formula (*Katsuyō Sanpō*) independently in 1683, somewhat around the same time as Jacob Bernoulli or earlier [1]. These numbers are important in combinatorics and number theory, and they also arise in the formulas of some special functions such as the Riemann zeta function. Jacob Bernoulli published this work in 1713 in *"Ars conjectandi"* [5]. These numbers are now popular as the Bernoulli numbers and they are associated with Bernoulli's polynomials.

The triangular numbers and their properties developed by the Pythagoreans 2000 years ago form the foundation of finding these Bernoulli numbers. The triangular numbers $1, 3, 6, 10, \dots$ were a sum of consecutive integers $1, 2, 3, 4, \dots$ It inspired the mathematicians of a later age to find the sum of powers of consecutive integers. Especially, the mathematicians of the 17^{th} and 18^{th} centuries studied such sums of finite and infinite series [6, 7]. According to Louis Saalschütz, the term Bernoulli's numbers was used for the first time by Abraham De Moivre (1667–1754) and also by Leonhard Euler (1707–1783) in 1755 [8–13]. Among the astonishing questions of these times was the Basel problem, which was proposed in 1644 by Pietro Mengoli and remained unsolved in his time [12]. Leonhard Euler extended the use of Bernoulli numbers and studied their properties in detail [1,13]. We are focusing on this particular direction.

2 Faulhaber's work and Bernoulli numbers

Consider a table where the table entry $(a_{ij}) = \binom{i}{j}$ forms the Pascal triangle when arranged in triangular form. This table is very useful for binomial expansions as well.

Table 1: Pascal triangle of binomial coefficients

$i \setminus j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0
5	1	5	10	10	5	1	0	0	0
6	1	6	15	20	15	6	1	0	0
7	1	7	21	35	35	21	7	1	0
8	1	8	28	56	70	56	28	8	1
9	1	9	36	84	126	126	84	36	9
10	1	10	45	120	210	252	210	120	45
11	1	11	55	165	330	462	462	330	165
12	1	12	66	220	495	792	924	792	495
13	1	13	78	286	715	1287	1716	1716	1287

Johann Faulhaber, a German mathematician, devoted considerable effort to finding formulas for the sums of powers of positive integers. His results, published in *Academia Algebrae* (1631), extended earlier work on triangular, square, and cubic sums to much higher powers. In particular, Faulhaber obtained explicit formulas for sums up to the 17^{th} power, and Donald Knuth (1993) noted that his tables implicitly encode formulas up to the 23^{rd} power [1, 14]. Faulhaber began with the triangular number $N = \frac{n(n+1)}{2}$

$$1^1 + 2^1 + 3^1 + 4^1 + \cdots + n^1 = N = \frac{n(n+1)}{2} \tag{1}$$

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = N^2 \tag{2}$$

$$1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \frac{4N^3 - N^2}{3} \tag{3}$$

$$1^7 + 2^7 + 3^7 + 4^7 + \cdots + n^7 = \frac{12N^4 - 8N^3 + 2N^2}{6} \tag{4}$$

$$1^{17} + 2^{17} + 3^{17} + 4^{17} + \cdots + n^{17} =$$

$$\frac{1280N^9 - 6720N^8 + 21120N^7 - 46880N^6 + 7291N^5 - 74220N^4 + 43404N^3 - 10851N^2}{45} \tag{5}$$

and so on.

For illustration, let us take $N = 10$, which implies that $n = 4$. By using this process, we can obtain Table 2.

So, from Equation (3), we have

$$1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \frac{4N^3 - N^2}{3}$$

$$\Rightarrow 1^5 + 2^5 + 3^5 + 4^5 = \frac{4 \times 10^3 - 10^2}{3} = 1300 \qquad \text{which is true.}$$

Table 2: Faulhaber's representation of N as triangular numbers in terms of positive integers n

N	1	3	6	10	15	21	28	36	45	55	66	78	91	105
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14

He claimed that these formulae make it very easy to compute finite sums of odd powers of positive integers. But it is difficult to find how he actually proved these formulae. In 1834, Carl Jacobi published a paper in which he provided a rigorous proof of Faulhaber's assertion [3, 4, 13–15]. Thus, Faulhaber expressed the sum of powers of the first n positive integers as [3]

$$S_p(n) = \sum_{k=1}^{n-1} k^p \tag{6}$$

i.e.,

$$\begin{aligned}
S_0(n) &= n & S_6(n) &= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} \\
S_1(n) &= \frac{n^2}{2} - \frac{n}{2} & S_7(n) &= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^5}{24} + \frac{n^2}{12} \\
S_2(n) &= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} & S_8(n) &= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{1}{30}n \\
S_3(n) &= \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4} & S_9(n) &= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{n^2}{12} \\
S_4(n) &= \frac{n^5}{5} - \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} & S_{10}(n) &= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66} \\
S_5(n) &= \frac{n^6}{6} - \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}
\end{aligned}$$

Here, the coefficient of n in $S_0(n)$ is 1, in $S_1(n)$ is $-\frac{1}{2}$, in $S_2(n)$ is $\frac{1}{6}$, in $S_3(n)$ is 0, in $S_4(n)$ is $-\frac{1}{30}$ and so on, which are known as Bernoulli numbers [13, 16]. In the form of continuity of Faulhaber's work, Jacob Bernoulli assumed $u = 2N = n^2 + n$. Then,

$$\begin{aligned}
\Sigma n &= \frac{1}{2}u = \frac{1}{2}A_0^1 u \\
\Sigma n^3 &= \frac{1}{4}u^2 = \frac{1}{4}A_0^2 u^2 + A_1^2 u \\
\Sigma n^5 &= \frac{1}{6}\left(u^3 - \frac{1}{2}u^2\right) = \frac{1}{6}\left(A_0^3 u^3 + A_1^3 u^2 + A_2^3 u\right) \\
\Sigma n^7 &= \frac{1}{8}\left(u^4 - \frac{4}{3}u^3 + \frac{2}{3}u^2\right) = \frac{1}{8}\left(A_0^4 u^4 + A_1^4 u^3 + A_2^4 u^2 + A_3^4 u\right)
\end{aligned}$$

and so on. He generalized and wrote the following expression

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left\{B_0 n^{m+1} - \binom{m+1}{1} B_1 n^m + \binom{m+1}{2} B_2 n^{m-1} + \cdots + (-1)^m \binom{m+1}{m} B_m n\right\} \tag{7}$$

Here, Jacob Bernoulli found the fixed values of $B_0, B_1, B_2, B_3, \ldots$ which are known as Bernoulli numbers. For illustration, the sum of the 4$^{\text{th}}$ power of positive integers up to 4 terms is given by:

$$\sum_{k=0}^{3} k^4 = \frac{1}{5}\left\{B_0 n^5 - \binom{5}{1} B_1 n^4 + \binom{5}{2} B_2 n^3 - \binom{5}{3} B_3 n^2 + \binom{5}{4} B_4 n^1\right\}$$

$$\sum_{k=0}^{3} k^4 = \frac{1}{5}\left\{n^5 - 5 \times \frac{1}{2}n^4 + 10 \times \frac{1}{6}n^3 + 5 \times \frac{-1}{30}n^1\right\}$$

Taking $n = 4$ in the right-hand side:

$$\sum_{k=0}^{3} k^4 = \frac{1}{5}\left\{4^5 - 5 \times \frac{1}{2}4^4 + 10 \times \frac{1}{6}4^3 + 5 \times \frac{-1}{30}4^1\right\} = 98$$

i.e., $1^4 + 2^4 + 3^4 = 98$, which is true.

3 Origin of Bernoulli numbers

Jacob Bernoulli deeply studied Faulhaber's formulae and the trends of the polynomial $S_p(n)$ defined by [3]

$$S_p(n) = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k} \tag{8}$$

$$= \frac{B_0}{0!} \frac{n^{p+1}}{(p+1)} + \frac{B_1}{1!} n^p + \frac{B_2}{2!} p n^{p-1} + \frac{B_3}{3!} p(p-1) n^{p-2} + \cdots + \frac{B_p}{1!} n$$

where B_k are numbers which are independent of p and are known as Bernoulli numbers. Then by identification with Faulhaber's formulae, Jacob Bernoulli concluded that the constant numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30},$$
$$B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0,$$
$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798}, \ldots$$

which are the Bernoulli numbers [5].

For illustration, these numbers work as claimed by Jacob Bernoulli. Consider $p = 4$ and $n = 4$:

$$S_4(4) = \sum_{k=0}^{3} k^4 = 1^4 + 2^4 + 3^4 = 98$$

Now, by using Equation (8):

$$S_4(4) = \sum_{k=0}^{4} \frac{B_k}{k!} \frac{4!}{(4+1-k)!} 4^{4+1-k}$$

$$= \frac{B_0}{0!} \times \frac{4^5}{5} + \frac{B_1}{1!} \times 4^4 + \frac{B_2}{2!} \times 4 \times 4^3 + \frac{B_3}{3!} \times 4 \times 3 \times 4^2 + \frac{B_4}{4!} \times 4 \times 3 \times 2 \times 4^1$$

$$= \frac{4^5}{5} - \frac{1}{2} \times 4^4 + \frac{1}{3} \times 4^3 - \frac{4}{30} = 98$$

The result is the same as the direct sum. So, the expressions are identical. Thus, it showed that the Bernoulli numbers are very useful in finding the sum of powers of positive integers. The original method for finding the Bernoulli numbers is computationally lengthy and it may take more time for higher Bernoulli numbers.

4 The Bernoulli generating function

In 1755, Euler explained the Bernoulli numbers as the coefficients of the exponential function [17–20]

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \tag{9}$$

$$\Rightarrow x = (e^x - 1)\left(\frac{B_0 x^0}{0!} + \frac{B_1 x^1}{1!} + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \frac{B_4 x^4}{4!} + \ldots\right)$$

$$\Rightarrow x = \left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(\frac{B_0 x^0}{0!} + \frac{B_1 x^1}{1!} + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \frac{B_4 x^4}{4!} + \ldots\right)$$

Multiplying both expressions on the right side and equating the coefficients of $x, x^2, x^3, \ldots$ on both sides yields

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \ldots$$

where $B_{2n+1} = 0$ for $n \geq 1$. As the value of n increases, the magnitude of the Bernoulli numbers B_{2n} grows rapidly. It is also observed that the denominators of every Bernoulli number B_{2n} $(n \geq 2)$ are divisible by 6. Furthermore, these numbers exhibit alternating signs such that $B_{4n} < 0$ and $B_{4n+2} > 0$ for $n \geq 1$ [7, 21–25].

5 The relation between Zeta function and Bernoulli numbers

The function $\sin x$ has zeroes at $x = 0, \pm\pi, \pm 2\pi, \pm 4\pi, \pm 6\pi, \dots$ Then Euler's infinite product representation for the sine function gives [26]

$$\sin x = kx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-4\pi)(x+4\pi)\cdots = kx(x^2-\pi^2)(x^2-4\pi^2)(x^2-16\pi^2)\dots \quad (10)$$

$$\Rightarrow \frac{\sin x}{x} = k(x^2-\pi^2)(x^2-4\pi^2)(x^2-16\pi^2)\dots \quad (11)$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = k(-\pi^2)(-4\pi^2)(-16\pi^2)\dots$$

$$\Rightarrow 1 = k(-\pi^2)(-4\pi^2)(-16\pi^2)\dots$$

$$\text{Therefore}, k = \frac{1}{(-\pi^2)(-4\pi^2)(-16\pi^2)\dots}$$

Substituting the value of k in Equation (11)

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\left(1-\frac{x^2}{25\pi^2}\right)\dots \quad (12)$$

Now, replacing x by πx in Equation (12), we get

$$\sin \pi x = \pi x\left(1-\frac{\pi^2x^2}{\pi^2}\right)\left(1-\frac{\pi^2x^2}{4\pi^2}\right)\left(1-\frac{\pi^2x^2}{9\pi^2}\right)\left(1-\frac{\pi^2x^2}{16\pi^2}\right)\dots$$

$$\frac{\sin \pi x}{\pi x} = (1-x^2)\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{16}\right)\left(1-\frac{x^2}{25}\right)\dots \quad (13)$$

Taking the logarithm on both sides of Equation (13), we get

$$\ln(\sin \pi x) = \ln \pi x + \sum_{k=1}^{\infty} \ln\left(1-\frac{x^2}{k^2}\right)$$

Differentiating both sides with respect to x, we get

$$\pi \cot \pi x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{1}{(1-\frac{x^2}{k^2})} \times \frac{2x}{k^2}$$

It is noted that $\frac{1}{1-\frac{x^2}{k^2}}$ must be the sum of an infinite geometric series with common ratio $\frac{x^2}{k^2}$ and first term 1. In this regard, we can write

$$\pi x \cot \pi x = 1 - 2\sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty}\left(\frac{x^2}{k^2}\right)^n\right) \times \frac{x^2}{k^2} \quad (14)$$

$$\pi x \cot \pi x = 1 - 2\sum_{k=1}^{\infty}\left\{\left(\frac{x^2}{k^2}\right)^1 + \sum_{n=1}^{\infty}\left(\frac{x^2}{k^2}\right)^{n+1}\right\} = 1 - 2\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{x^2}{k^2}\right)^n\right\}$$

But,

$$\sum_{k=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{x^2}{k^2}\right)^n\right\} = \sum_{k=1}^{\infty}\left(\frac{x^2}{k^2}\right)^1 + \sum_{k=1}^{\infty}\left(\frac{x^2}{k^2}\right)^2 + \sum_{k=1}^{\infty}\left(\frac{x^2}{k^2}\right)^3 + \dots$$

$$= x^2\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^1 + x^4\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^2 + x^6\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)^3 + \dots$$

$$= x^2\zeta(2) + x^4\zeta(4) + x^6\zeta(6) + x^8\zeta(8) + \cdots = \sum_{n=1}^{\infty} x^{2n}\zeta(2n)$$

Substituting this value in Equation (14), one can yield

$$\pi x \cot \pi x = 1 - 2\sum_{n=1}^{\infty} x^{2n}\zeta(2n) \tag{15}$$

Again, from Euler's effort:

$$\cot x = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n(2)^{2n}(x)^{2n-1}}{(2n)!} \tag{16}$$

Replacing x by πx and multiplying both sides of Equation (16) by πx itself, we get

$$\pi x \cot \pi x = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n(2)^{2n}(\pi x)^{2n}}{(2n)!} \tag{17}$$

So, from Equations (15) and (17) we have

$$1 - 2\sum_{n=1}^{\infty}(x)^{2n}\zeta(2n) = \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n(2)^{2n}(\pi x)^{2n}}{(2n)!}$$

$$\Rightarrow 1 - 2\sum_{n=1}^{\infty}(x)^{2n}\zeta(2n) = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^n(2)^{2n}(\pi x)^{2n}}{(2n)!}$$

$$\Rightarrow \sum_{n=1}^{\infty}(x)^{2n}\zeta(2n) = \frac{-1}{2}\sum_{n=1}^{\infty} \frac{B_{2n}(-1)^n(2)^{2n}(\pi x)^{2n}}{(2n)!}$$

$$\Rightarrow \sum_{n=1}^{\infty}(x)^{2n}\zeta(2n) = \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^{n+1}(2)^{2n-1}(\pi)^{2n}(x)^{2n}}{(2n)!}$$

Equating the coefficients of $(x)^{2n}$ on both sides, one can obtain

$$\zeta(2n) = \frac{B_{2n}(-1)^{n+1}(2)^{2n-1}(\pi)^{2n}}{(2n)!}$$

$$\text{Therefore,} \zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!} \tag{18}$$

which is the required relation between the Riemann zeta function and Bernoulli numbers [5].

If $n = 1, 2, 3, 4, 5, \ldots$ in (18), then $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, $\zeta(8) = \frac{\pi^8}{9450}$, $\zeta(10) = \frac{\pi^{10}}{93555}, \ldots$ [27–29]. Thus, each $\zeta(2n)$ is a rational multiple of π^{2n}, where the rational coefficients are the Bernoulli numbers. The generating function $\frac{x}{e^x-1}$ converges for $|x| < 2\pi$, which is why Bernoulli numbers appear naturally in many analytic expansions. Euler's evaluation of these special zeta values is a fundamental result in analytic number theory [30].

6 Conclusion

This paper explored the relationship between sums of powers of positive integers and the Bernoulli numbers. The study examined the contributions of Johann Faulhaber, Jacob Bernoulli, and Leonhard Euler, highlighting the historical development of formulas for power sums and the emergence of coefficients now known as Bernoulli numbers. In particular, Euler's systematic derivation and interpretation of these numbers were discussed. This study demonstrates that Bernoulli numbers possess a rich mathematical structure, with applications extending beyond finite sums to analysis and number theory. Furthermore, this study examined the connection between the values of the Riemann zeta function and Bernoulli numbers, expressed as $\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$. This relation illustrates how Bernoulli numbers provide a bridge between discrete summation formulas and continuous analytic structures. Future studies may focus on Euler numbers, Euler polynomials, the role of the zeta function in summability theory, the application of Bernoulli numbers in the Euler-Maclaurin summation formula, and other properties of Bernoulli numbers.

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Conflict of Interest

There is no conflict to declare.

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