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# Analytical solution framework for fractional differential equations using Laplace transforms and Mittag-Leffler functions

**Suresh Bishwokarma[1], Shankar Pariyar[2*], Jeevan Kafle[1]**
[1] *Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.*
[2] *Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu, Nepal.*

*Corresponding author: Email: shankar.pariyar@trc.tu.edu.np

**Abstract**

*Fractional differential equations have emerged as an important mathematical tool for modeling dynamical phenomena in which memory and hereditary effects play a significant role. Unlike classical differential equations, their fractional-order counterparts involve nonlocal operators, making the construction of exact analytical solutions considerably more challenging. This paper presents a unified analytical framework based on the combined use of Mittag–Leffler functions and the Laplace transform for obtaining closed-form solutions of fractional differential equations. The study first reviews the Grünwald–Letnikov, Riemann–Liouville, and Caputo definitions of fractional derivatives together with their essential mathematical properties. The analytical characteristics of the one-parameter and two-parameter Mittag–Leffler functions are then examined, and the corresponding Laplace transform relations required for solution development are established. These theoretical results are subsequently employed to derive exact solutions for representative fractional differential equations. The analysis confirms that Mittag–Leffler functions naturally generalize the exponential function and accurately characterize the behavior of fractional-order systems. In particular, for the one-term Caputo fractional differential equation* ${}_0^C D_x^\alpha y(x) + \lambda y(x) = h(x)$*, an analytical solution is obtained in terms of the Mittag–Leffler function and its associated convolution integral. Furthermore, the integration of Laplace transform techniques with Mittag–Leffler functions provides a systematic and rigorous procedure for solving both homogeneous and non-homogeneous fractional models. The proposed framework contributes to the theoretical analysis of fractional differential equations and offers a practical foundation for future investigations in applied mathematics, science, and engineering.*

**Keywords**
Fractional calculus, fractional differential equations, Laplace transform, Mittag–Leffler functions, analytical solutions.

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## 1 Introduction

Fractional calculus has become an important branch of modern mathematical analysis due to its capability to represent dynamical systems governed by memory and hereditary mechanisms [1, 2]. By extending the concepts of differentiation and integration from integer orders to arbitrary real or complex orders, it provides a broader mathematical framework than classical calculus for describing nonlocal processes [3, 4]. This generalization has led to the widespread use of fractional differential equations (FDEs) in modeling phenomena whose present states depend not only on current conditions but also on their past evolution. Such memory-dependent behavior frequently arises in scientific and engineering applications, where conventional integer-order models often fail to capture the observed dynamics with sufficient accuracy [2, 5, 6].

The concept of fractional differentiation dates back to the question raised by Leibniz concerning the meaning of a derivative of non-integer order, which later stimulated the development of fractional calculus as a distinct mathematical discipline [3, 7]. During the following centuries, the mathematical foundations of fractional calculus were established through the pioneering contributions of Liouville, Riemann, Grünwald, Letnikov, and Caputo, leading to several widely used definitions of fractional derivatives [2, 8, 9]. Among these, the Grünwald–Letnikov, Riemann–Liouville, and Caputo operators have become the principal formulations for theoretical investigations and practical applications because each possesses distinct analytical properties and is suitable for different classes of fractional differential equations [10, 11].

In recent years, fractional differential equations (FDEs) have been widely employed to model anomalous diffusion, viscoelastic behavior, transport processes, biological systems, electrical circuits, and other complex phenomena characterized by memory-dependent dynamics [12–15]. Their increasing importance has motivated extensive research on analytical, semi-analytical, and numerical solution techniques [16, 17]. Although a variety of numerical methods have been developed for solving FDEs, analytical solutions remain of particular interest because they provide exact representations of the underlying mathematical models, serve as reliable benchmarks for validating numerical algorithms, and offer deeper insight into the qualitative behavior of fractional-order systems [18–20].

Among the many analytical techniques proposed for fractional differential equations, the Laplace transform occupies a central position because it converts fractional differential operators into algebraic expressions in the transform domain, allowing the solution process to be carried out in a systematic manner while incorporating initial conditions naturally [20–22]. Its effectiveness is closely linked to the Mittag–Leffler function, introduced by Gösta Mittag–Leffler in 1903 [23]. In fractional calculus, this function plays a role analogous to that of the exponential function in classical differential equations and appears naturally in exact solutions of a wide range of linear fractional models [11, 24, 25]. Consequently, Mittag–Leffler functions have become indispensable in the analytical treatment of FDEs, particularly when combined with Laplace-transform-based solution procedures and related integral transform techniques [26–28].

Although the literature on fractional calculus has expanded considerably, existing studies typically concentrate on one aspect of the subject. Some investigations focus on developing new fractional derivative operators and their theoretical properties [29], whereas others derive analytical or numerical solutions for particular classes of fractional differential equations [30, 31]. Recent contributions follow a similar direction. Vatsala and Pageni [32] established analytical solutions of linear Caputo equations with fractional initial conditions, while Abolhassanifar et al. [33] incorporated Mittag–Leffler stability analysis into the Laplace-transform framework for fuzzy Caputo systems. Al-Refai et al. [34] investigated equations involving a modified Mittag–Leffler kernel under Dirac delta forcing, Cruz-López and Espinosa-Paredes [35] derived analytical solutions of fractional neutron point kinetic equations using Laplace transforms together with Mittag–Leffler functions, and Duan et al. [36] obtained solutions of a two-term Caputo oscillation equation through a combination of Mittag–Leffler series and Laplace-transform techniques. While these studies clearly demonstrate the effectiveness of both Mittag–Leffler functions and Laplace transforms, their analyses are primarily directed toward specific mathematical models. A comprehensive framework that systematically integrates the principal fractional derivative operators, Mittag–Leffler functions, and Laplace-transform methodology within a single analytical development remains comparatively limited.

The present work is intended to provide such an integrated framework. Rather than examining an isolated equation, it develops the Grünwald–Letnikov, Riemann–Liouville, and Caputo fractional derivatives alongside the one- and two-parameter Mittag–Leffler functions and their corresponding Laplace-transform identities within a unified analytical setting. These mathematical tools are then applied to a sequence of representative fractional differential equations of increasing complexity to obtain exact solutions. Whenever possible, solutions derived through the Mittag–Leffler approach are independently verified by the Laplace-transform method, thereby demonstrating the consistency of the proposed methodology. By establishing explicit connections among fractional operators, special functions, and integral transforms, the paper provides a coherent analytical framework that can be used as a foundation for future theoretical investigations and practical applications of fractional differential equations.

## 2 Preliminaries

**Definition 1** The Gamma function $\Gamma(n)$ is given by [37]

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$$

and the integral is convergent in the right half-plane $Re(n) > 0$.

**Definition 2** For two positive real numbers $m$ and $n$, [37]

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

**Definition 3** The one-parameter Mittag–Leffler function (MLF) [23],

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

A more general form is obtained in two parameter MLF

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

where $(\alpha, \beta \in \mathbb{C})$ and $(\mathrm{Re}(\alpha) > 0)$. This expression is called the two-parameter Mittag–Leffler function [23]. For the particular choice $(\alpha = 1)$ and $(\beta = 1)$, $E_{1,1}(z) = e^z$.

## 3 Fractional derivatives and integrals

**Definition 4** The Grünwald–Letnikov (GL) Fractional Derivative [3, 4] on the forward difference derivative, by considering $f(x)$ continuous function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

By using the principle of induction,

$$f^n(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - jh)$$

where, $n$ is an integer. We now rewrite the expression in a form that extends its fractional order where $\alpha$ is any number and by changing limit $h = \frac{x-a}{n}, h \implies 0, n \implies \infty, a < x \implies \frac{1}{h} = \frac{n}{x-a}$

Then our definition becomes, $f^{(\alpha)}(x) = \lim_{n\to\infty} \left(\frac{1}{h}\right)^\alpha \sum_{j=0}^{n} (-1)^j \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)j!} f\left(x - jh\right)$ which is GL Fractional Derivative [4] and simply known as GL Fractional Derivative. This definition is quite complicated which involves limits and it make sense only when limit of the function exists. In case the limit of the function exist but it is not an easy job to evaluate the limit for every function. So we have to take some modification for this definition.

let us consider a expression

$$f^\alpha(x) = \lim_{n \to \infty} \frac{1}{h^\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} f(x - jh)$$

If we plug $-\alpha$ in place of $\alpha$,

$$f^{-\alpha}(x) = \lim_{n \to \infty} h^\alpha \sum_{j=0}^{n} \frac{\Gamma(\alpha + j)}{j!\Gamma(\alpha)} f(x - jh)$$

Equivalently, [9]

$$I^n(f) = \frac{1}{\Gamma(n)} \int_a^x (x-u)^{n-1} f(u)\, du$$

**Definition 5** Replacing the integer $n$ with an arbitrary real number $\alpha$ gives

$$_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u)\, du, \qquad \alpha > 0.$$

which is known as Riemann–Liouville (RL) fractional integral [9] of order $\alpha$.

Some properties:

- Linearity: $_aI_x^\alpha(cf) = c_a I_x^\beta(f)$ and $_aI_x(f \pm g) = {_aI_x}(f) \pm {_aI_x}(g)$
- $_aI_x^\alpha[_aI_x^\alpha] = {_aI_x^{\alpha+\beta}}(f) = {_aI_x^\beta}[_aI_x^\alpha(f)]$

### 3.1 RL fractional integral of elementary functions

- **Integration of Constants** $y(x) = C$

  By the definition of Riemann-Liouville [9] Fractional Integral,

  $$_a\mathcal{I}_x^\alpha(C) = \frac{C(x-a)^\alpha}{\Gamma(\alpha+1)}$$

  In order to match the result with the classical calculus we must put the base of the integral $a = 0$ with $\alpha = 1$. Then the integral becomes,

  $$_0\mathcal{I}_x^\alpha(C) = \frac{Cx^\alpha}{1.\Gamma(1)} = Cx$$

. **Integration of** $y(x) = x^n$
Using the RL definition of the fractional integral [9],

$$ {}_a\mathcal{I}_x^\alpha(x^n) = \frac{x^{\alpha+n}}{\Gamma(\alpha)} \int_{\frac{a}{x}}^{1} (1-u)^{\alpha-1} u^{(n+1)-1}\, du $$

If we put the base point $a = 0$ then $\frac{a}{x} = 0$ in the above integral, Thus, the integral involves the Beta function,

$$ {}_0\mathcal{I}_x^\alpha(x^n) = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} x^{\alpha+n} $$

Taking $\alpha = 1$ and $n = 1$ in the above formula gives the first-order integral of $x^n$, which can be written as

$$ {}_0\mathcal{I}_x^1(x^1) = \frac{x^2}{2} $$

This result matches with the result of classical calculus.

. **Integration of** $f(x) = e^{kx}$
By the definition of the RL fractional integral [9],

$$ {}_a\mathcal{I}_x^\alpha(e^{kx}) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{kt}\, dt. $$

To simplify this integral, take $x - t = u$, so that $dt = -du$. With this change of variable, the lower limit $t = a$ becomes $u = x-a$, while the upper limit $t = x$ becomes $u = 0$ Now,

$$ {}_a\mathcal{I}_x^\alpha(e^{kx}) = \frac{e^{kx}}{\Gamma(\alpha)} \int_0^{x-a} u^{\alpha-1} e^{-ku}\, du. $$

Next, set $ku = y$, which gives $du = \frac{dy}{k}$. Under this substitution, $u = 0$ corresponds to $y = 0$, and $u = x - a$ corresponds to $y = k(x - a)$. Hence,

$$ {}_a\mathcal{I}_x^\alpha(e^{kx}) = \frac{e^{kx}}{\Gamma(\alpha)k^\alpha} \int_0^{k(x-a)} e^{-y} y^{\alpha-1}\, dy. $$

For the computation of this integral, the base point must be chosen appropriately. The above integral resembles the Gamma function except for its upper limit. When $a \to -\infty$, then we have $k(x-a) \to \infty$, for $k > 0$. Therefore,

$$ {}_{-\infty}\mathcal{I}_x^\alpha(e^{kx}) = \frac{e^{kx}}{\Gamma(\alpha)k^\alpha} \int_0^{\infty} e^{-y} y^{\alpha-1}\, dy = \frac{e^{kx}}{k^\alpha}. $$

If we put $\alpha = 1$ and $k = 1$, then

$$ {}_{-\infty}\mathcal{I}_x^1(e^x) = \frac{e^x}{1} = e^x, $$

which agrees with the result from classical calculus.

. **Integration of** $y(x) = sinx$
Using the Riemann–Liouville fractional integral

$$ {}_a\mathcal{I}_x^\alpha(\sin x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \sin t\, dt. $$

we get,

$$ {}_a\mathcal{I}_x^\alpha(\sin x) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_a^x (x-t)^{\alpha-1} t^{2k+1}\, dt. $$

For $a = 0$, using the Beta-function identity, we get

$$ \boxed{ {}_0\mathcal{I}_x^\alpha(\sin x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1+\alpha}}{\Gamma(2k+\alpha+2)} }. $$

. **Integration of** $f(x) = \cos x$
Applying the RL fractional integral and $\alpha = 1$,

$$ {}_0\mathcal{I}_x^1(\cos x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin x. $$

**Definition 6** Let $f$ be a continuous function and let $\alpha > 0$. Suppose that $n$ is the least integer greater than $\alpha$, so that $\alpha \epsilon \mathbb{R}^+$ and $n - 1 < \alpha < n$.

Then the RL fractional derivative of order $\alpha$ is defined by [3]

$$ {}_aD_x^\alpha(f) = \frac{d^n}{dx^n}\left[ {}_aI_x^{n-\alpha} f(x) \right] $$

## 3.2 RL fractional derivatives of some elementary functions

. **Derivative of constant f(x)=C**

$$ {}_a^{RL}D_x^\alpha(C) = C \cdot \frac{1}{\Gamma(1-\alpha)(x-a)^\alpha} \neq 0 $$

As we can see, this outcome is not always zero.This outcome is in conflict with the classical calculus conclusion. This illustrates a key limitation of the RL fractional derivative. By reversing the sign in the order of the integral, one can derive the RL fractional derivative directly from the corresponding RL fractional integral.

. **RL Fractional Derivatives of** $y(x) = x^n$

$$ {}_a^{RL}D_x^\alpha(x^n) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} $$

. **RL Fractional Derivatives of** $y(x) = e^{kx}$

$$ {}_a^{RL}D_x^\alpha(e^{kx}) = k^\alpha e^{kx} $$

. **RL Fractional Derivatives of** $y(x) = \sin(kx)$

$$^{RL}_{a}D^{\alpha}_{x}(\sin(kx)) = k^{\alpha}\sin\left(kx + \frac{\alpha\pi}{2}\right)$$

. **RL Fractional Derivatives of** $y(x) = \cos(kx)$

$$^{RL}_{a}D^{\alpha}_{x}(\cos(kx)) = k^{\alpha}\cos\left(kx + \frac{\alpha\pi}{2}\right)$$

We see there is some drawback of RL fractional derivative, so to overcome the drawback of this RL definition there is a another definition of fractional derivative called Caputo Fractional Derivative(CFD) [38].

**Definition 7** Let $f$ be a function of class $\mathcal{C}$ and let $\alpha > 0$.

$$_{a}D^{\alpha}_{x}(f) = _{a} I^{n-\alpha}_{x}\left[\frac{d^n}{dx^n}f(x)\right]$$

is known as CFD [38] where $\alpha\epsilon\mathbb{R}^{+}, n-1<\alpha<n$.

Next, we consider the fractional derivative of the constant function $f(x) = C$, using the CFD definition [38].
$^{C}_{a}D^{\alpha}_{x}(C) = {}_{a}I^{(n-\alpha)}_{x}\left\{\frac{d^n}{dx^n}\cdot C\right\} = 0$ which is matched to classical derivative.

### 3.3 CFDs of some elementary functions

. CFD [38] of $f(x) = x^{\beta}; \beta > -1$

$$^{C}_{a}D^{\alpha}_{x}\left[x^{\beta}\right] = {}_{a}I^{n-\alpha}_{x}\cdot\frac{d^n}{dx^n}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\cdot x^{\beta-\alpha}$$

. CFD of exponential function $e^{\lambda x}$ [38]

$$\begin{aligned}^{C}_{a}D^{\alpha}_{x}\left[e^{\lambda x}\right] &= {}_{a}I^{n-\alpha}_{x}\left(\lambda^n e^{\lambda x}\right)\\ &= \lambda^n x^{n-\alpha}\cdot E_{1,\,n-\alpha+1}(\lambda x)\end{aligned}$$

. CFD of $y(x) = \sin(\lambda x)$ [38]

$$^{C}_{0}D^{\alpha}_{x}\left[\sin\lambda x\right] = \frac{1}{2i}\left[^{C}_{0}D^{\alpha}_{x}(e^{i\lambda x}) - {}^{C}_{0}D^{\alpha}_{x}(e^{-i\lambda x})\right]$$

Using $^{C}_{0}D^{\alpha}_{x}\left[e^{\lambda x}\right] = \lambda^n x^{n-\alpha}E_{1,n-\alpha+1}(\lambda x)$:

$$= \frac{1}{2i}(i\lambda)^n x^{n-\alpha}\left[E_{1,n-\alpha+1}(\lambda i x) - (-1)^n E_{1,n-\alpha+1}(-i\lambda x)\right]$$

. CFD of $y(x) = \cos(\lambda x)$ [38]

$$^{C}_{0}D^{\alpha}_{x}\left[\cos\lambda x\right] = \frac{1}{2}(i\lambda)^n x^{n-\alpha}\left[E_{1,n-\alpha+1}(i\lambda x) + (-1)^n E_{1,n-\alpha+1}(-i\lambda x)\right]$$

## 4 Laplace transform (LT)

**Definition 8** Let $f(t)$ be a continuous, single-valued function defined on the interval $(0,\infty)$ and of exponential order. Its Laplace transform, denoted by $F(s)$, is given by the following integral [22]

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st}f(t)dt \quad s>0$$

for those values of $s$ where the integral converges.
Some Useful Results

a) If $f(t) = 1$, then $F(s) = \frac{1}{s}$

b) If $f(t) = t$, then $F(s) = \frac{1}{s^2}$

c) If $f(t) = t^n$, then $F(s) = \frac{\Gamma(n+1)}{s^{n+1}}$

d) If $f(t) = e^{at}$, then $F(s) = \frac{1}{s-a}$

**Theorem 1 (Convolution Theorem [4])** If $F(s)$ and $G(s)$ are the Laplace transform of $f(x)$ and $g(x)$ respectively then $F(s)G(s)$ is the Laplace transform of

$$\int_0^x f(x-u)g(u)du.$$

## LT of MLF

The LT of $x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})$ as calculating the LT of $E_{\alpha,\beta}(x)$ only become quite complicated and it involves summation notation. So we modify the MLF into $x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})$ whose LT is in closed form.

$$\mathcal{L}[x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})] = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$$

Put $\alpha = 1, \beta = 1$

$$\mathcal{L}[E_{1,1}(x)] = \frac{1}{s-1} = \mathcal{L}[e^{x}]$$

## LT of RL fractional derivative

$$\mathcal{L}\left\{{}_{0}^{RL}D_{x}^{\alpha} f(x)\right\} =$$
$$s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{k}\, {}_{0}^{RL}D_{x}^{\alpha-k-1} f(0)$$

## LT of RL fractional integral

$\mathcal{L}[{}_{0}\mathcal{I}_{x}^{\alpha} f(x)] = s^{-\alpha}F(s)$

## LT of CFD

$$\mathcal{L}\left\{{}_{0}^{C}D_{x}^{\alpha} f(x)\right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$$

### 5 Fractional differential equations

**Definition 9** A fractional differential equation (FDE) is an equation involving derivatives of arbitrary order $\alpha > 0$, represented by $\frac{d^{\alpha}}{dx^{\alpha}}$, where $\alpha$ need not be an integer. Thus, FDEs constitute a natural generalization of ordinary differential equations to fractional orders.

In general, a fractional differential equation may be classified as linear or nonlinear depending on whether the unknown function $y(x)$ and its fractional derivatives appear linearly or nonlinearly in the equation [2, 5, 39, 40].

**Definition 10** The equation [2]

$$D_{x}^{\alpha_m} y(x) + \sum_{k=1}^{m-1} a_k D_{x}^{\alpha_k} y(x) + a_0 y(x) = f(x)$$

where $\alpha_m > \alpha_{m-1} > \cdots > \alpha_1 > \alpha_0 \geq 0$ and $a_k \in \mathbb{R}$, is called a RL-type fractional linear DE [2]. The corresponding initial conditions for this FDE are specified as follows:

$$D_{x}^{\alpha_m - k} y(0) = b_k\,; \qquad k = 1, 2, \ldots, m-1$$
$$D_{x}^{-(m-\alpha)} y(0) = b_m$$

Similarly, the equation

$${}^{C}D_{x}^{\alpha_m} y(x) + \sum_{k=1}^{m-1} a_k\, {}^{C}D_{x}^{\alpha_k} y(x) + a_0 y(x) = f(x)$$

is called a CFD-type FDE, with initial conditions expressed in the classical integer-order form

$$\left.\frac{d^{k}y}{dx^{k}}\right|_{x=0} = b_k.$$

The form of the initial conditions clearly differentiates RL-based FDEs from CFD-based FDEs. Since the CFD allows the use of standard integer-order initial data, it is particularly convenient for practical applications [38].

**Definition 11** An FDE is considered nonlinear if the unknown function $y(x)$ or its fractional derivatives appear nonlinearly. In general, a nonlinear fractional differential equation takes the form

$$D_x^{\alpha} y(x) = F\big(x,\, y(x),\, D_x^{\alpha_1} y(x),\, D_x^{\alpha_2} y(x),\, \ldots, D_x^{\alpha_{m-1}} y(x)\big)$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < \alpha$ and $F$ is a nonlinear function of its arguments [2, 5].

## 5.1 Analytical treatment of FDEs using MLFs and LT methods

The analytical treatment begins by assuming a Mittag–Leffler function as the candidate solution of the fractional differential equation. The proposed solution is then substituted into the governing equation to establish its validity. An independent derivation based on the Laplace transform is subsequently carried out to verify the obtained result. Finally, the fractional solution is examined in the limiting case ($\alpha = 1$), where it reduces to the corresponding classical solution [4, 24, 27].

### 5.1.1 One-term homogeneous FDE via the Mittag-Leffler function involving CFD

**Example 1** *Solve the one-term fractional differential equation via MLF*

$$\frac{d^{\alpha} y}{dx^{\alpha}} \;=\; Ay.$$

We assume the solution in the form

$$y \;=\; E_{\alpha}(a\,x^{\alpha}) \;=\; \sum_{n=0}^{\infty} \frac{a^n\, x^{\alpha n}}{\Gamma(\alpha n + 1)}.$$

Since $\dfrac{d^{\alpha}}{dx^{\alpha}} = {}_0^C D_x^{\alpha}$, substituting into ${}_0^C D_x^{\alpha}[y] = Ay$ after applying the term-by-term fractional derivative, using ${}_0^C D_x^{\alpha}[x^n]$

$$\sum_{n=0}^{\infty} \frac{a^n\, x^{\alpha n - \alpha}}{\Gamma(\alpha n - \alpha + 1)} \;=\; A \sum_{n=0}^{\infty} \frac{a^n\, x^{\alpha n}}{\Gamma(\alpha n + 1)}.$$

Rewriting after taking $\alpha$ common on the left:

$$\sum_{n=0}^{\infty} \frac{a^n\, x^{(n-1)\alpha}}{\Gamma\big((n-1)\alpha + 1\big)} \;=\; A \sum_{n=0}^{\infty} \frac{a^n\, x^{n\alpha}}{\Gamma(n\alpha + 1)}.$$

Replacing the dummy variable $n - 1 = k \Rightarrow n = k + 1$: and separating the $k = -1$ term on the left:

$$\frac{a^0\, x^{-\alpha}}{\Gamma(-\alpha + 1)} + \sum_{k=0}^{\infty} \frac{a^{k+1}\, x^{k\alpha}}{\Gamma(k\alpha + 1)} \;=\; A \sum_{k=0}^{\infty} \frac{a^k\, x^{k\alpha}}{\Gamma(\alpha k + 1)}$$
$$\Rightarrow \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=0}^{\infty} \left[ \frac{a^{k+1}}{\Gamma(\alpha k + 1)} - \frac{A\, a^k}{\Gamma(\alpha k + 1)} \right] x^{k\alpha} = 0.x^{-\alpha} + 0.x^{k\alpha}$$

Comparing coefficients:

- Coefficient of $x^{-\alpha}$: $\quad \dfrac{1}{\Gamma(1-\alpha)} = 0.$
- Coefficient of $x^{k\alpha}$ for $k \geq 0$:

$$\frac{a^{k+1}}{\Gamma(\alpha k + 1)} - \frac{A\, a^k}{\Gamma(\alpha k + 1)} = 0 \implies a^{k+1} = A\, a^k, \quad k = 0, 1, 2, \ldots$$

From the recurrence $a^{k+1} = A\,a^k$, iterating:

$$k=0:\; a^1 = A\,a^0 = A, \quad k=1:\; a^2 = A^2, \quad k=2:\; a^3 = A^3, \quad \ldots \quad a^n = A^n.$$

Since we assumed $y = E_\alpha(ax^\alpha) = \sum_{n=0}^{\infty} \frac{a^n x^{n\alpha}}{\Gamma(\alpha n+1)}$, substituting $a^n = A^n$:

$$\boxed{y \;=\; E_\alpha(A\,x^\alpha) \;=\; \sum_{n=0}^{\infty} \frac{A^n\,x^{n\alpha}}{\Gamma(\alpha n+1)}}$$

is the solution of the one-term FDE.
Verification with Classical Solution when $(\alpha = 1)$

$$\text{therefore,} y = E_1(Ax) = e^{Ax}.$$

The classical ODE $\frac{dy}{dx} = Ay$ has solution $y = e^{Ax}$, which matches.

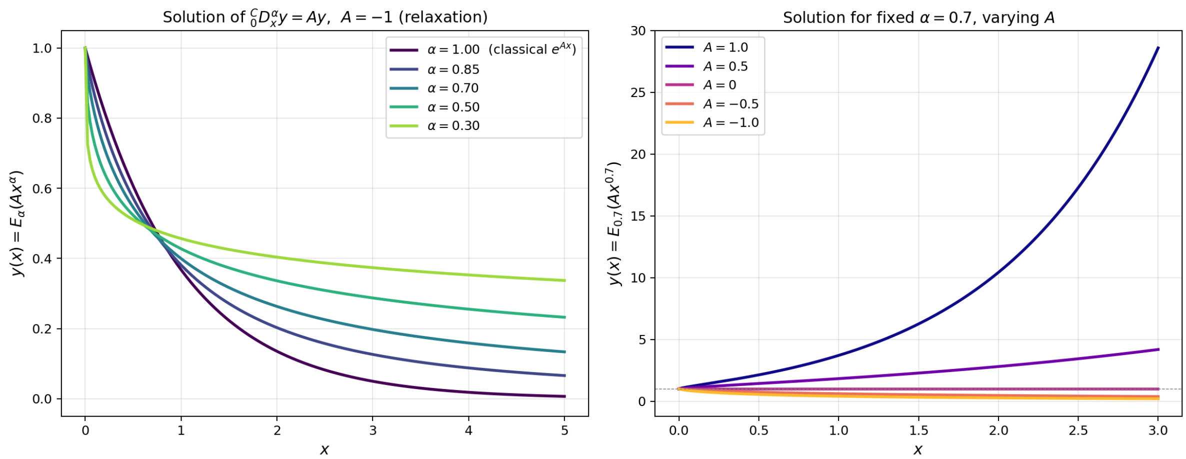


Figure 1: Solution $y(x) = E_\alpha(Ax^\alpha)$ of the one-term FDE ${}^C_0D^\alpha_x y = Ay$ for several parameter choices, verified against the classical solution $e^{Ax}$ at $\alpha = 1$.

Figure 1 illustrates the behaviour of the solution $y(x) = E_\alpha(Ax^\alpha)$ obtained above.

**Left panel:** $y(x) = E_\alpha(Ax^\alpha)$ for fixed $A = -1$, varying $\alpha$. As $\alpha \to 1$, the curve reduces to the familiar exponential relaxation $e^{-x}$; for smaller $\alpha$, the curve exhibits the classic fractional signature — a faster initial decay followed by a long, slow (power-law) tail, in contrast to the pure exponential decay of the classical case.

**Right panel:** Fixed $\alpha = 0.7$, varying $A$. A positive $A$ produces Mittag-Leffler growth (the fractional analogue of $e^{Ax}$ blowing up), $A = 0$ gives the constant solution $y = 1$, and negative $A$ produces decay.

**Verification:** At $\alpha = 1$, $E_1(-x)$ matches $e^{-x}$ to machine precision, confirming both the boxed result of the derivation and its correct reduction to the classical solution in the integer-order limit.

#### 5.1.2 Two-term homogeneous FDE via the MLF involving CFD

**Example 2** *Solve the two-term fractional differential equation via the MLF*

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} + \frac{d^\alpha y}{dx^\alpha} - 2y \;=\; 0.$$

Assume the solution of the form

$$y \;=\; E_\alpha(a\,x^\alpha) \;=\; \sum_{n=0}^{\infty} \frac{a^n\,x^{\alpha n}}{\Gamma(\alpha n+1)}.$$

Since $\dfrac{d^\alpha}{dx^\alpha} = {}_0^C D_x^\alpha$, substituting into the equation:

$$ {}_0^C D_x^{2\alpha}\left[\sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha n}}{\Gamma(\alpha n+1)}\right] + {}_0^C D_x^{\alpha}\left[\sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha n}}{\Gamma(\alpha n+1)}\right] - 2\sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha n}}{\Gamma(\alpha n+1)} \;=\; 0. $$

We know that ${}_0^C D_x^\alpha[x^n] = \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\,x^{n-\alpha}$ and ${}_0^C D_x^{2\alpha}[x^n] = \dfrac{\Gamma(n+1)}{\Gamma(\alpha n-2\alpha+1)}\,x^{\alpha n-2\alpha}$.
Applying term-by-term

$$ \sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha(n-2)}}{\Gamma\big(\alpha(n-2)+1\big)} + \sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha(n-1)}}{\Gamma\big(\alpha(n-1)+1\big)} - 2\sum_{n=0}^{\infty}\frac{a^n\,x^{\alpha n}}{\Gamma(\alpha n+1)} \;=\; 0. $$

Re-indexing the sums:
First term: $n-2=k \;\Rightarrow\; n=k+2$
Second term: $n-1=k \;\Rightarrow\; n=k+1$
Third term: $n=k$

$$ \sum_{k=0}^{\infty}\frac{a^{k+2}\,x^{\alpha k}}{\Gamma(\alpha k+1)} + \sum_{k=0}^{\infty}\frac{a^{k+1}\,x^{\alpha k}}{\Gamma(\alpha k+1)} - 2\sum_{k=0}^{\infty}\frac{a^{k}\,x^{\alpha k}}{\Gamma(\alpha k+1)} \;=\; 0. $$
$$ \sum_{k=0}^{\infty}\left[a^{k+2}+a^{k+1}-2a^k\right]\frac{x^{\alpha k}}{\Gamma(\alpha k+1)} \;=\; 0. $$

Comparing coefficients (setting the bracket to zero):
$a^{k+2}+a^{k+1}-2a^k \;=\; 0.$
Factoring out $a^k$ (noting $a^k \neq 0$ to avoid the trivial solution):
$a^k\big(a^2+a-2\big)=0 \;\Longrightarrow\; a^2+a-2=0.$
Solving the characteristic equation:
$a^2+a-2=0 \;\Longrightarrow\; (a+2)(a-1)=0 \;\Longrightarrow\; a=-2 \quad\text{or}\quad a=1$. Since we assumed $y=E_\alpha(a\,x^\alpha)$, the two independent solutions are
$y_1 \;=\; E_\alpha(x^\alpha), \qquad y_2 \;=\; E_\alpha(-2x^\alpha).$
By the principle of superposition,
the general solution is $\boxed{y \;=\; c_1\,E_\alpha(x^\alpha)+c_2\,E_\alpha(-2x^\alpha)}$ where $c_1$ and $c_2$ are arbitrary constants.
Verification with Classical Solution ($\alpha=1$)
Setting $\alpha=1$, the FDE reduces to the classical ODE:

$$ \frac{d^2y}{dx^2}+\frac{dy}{dx}-2y \;=\; 0. $$

The characteristic equation is $D^2+D-2=0$, giving roots $D=1$ and $D=-2$. The classical solution is $y \;=\; C_1\,e^x+C_2\,e^{-2x}$.
Using $E_1(z)=e^z$, the Mittag-Leffler solution becomes

$$ y \;=\; c_1\,E_1(x^1)+c_2\,E_1(-2x^1) \;=\; c_1\,e^x+c_2\,e^{-2x}, $$

which matches the classical solution exactly.

Figure 2 illustrates the behaviour of the general solution $y(x)=c_1E_\alpha(x^\alpha)+c_2E_\alpha(-2x^\alpha)$ obtained above.

**Left panel:** $y(x)$ for fixed $c_1=c_2=1$, varying $\alpha$.For $\alpha=1$, the fractional solution coincides with its classical counterpart $e^x+e^{-2x}$, a decaying transient superposed on exponential growth. As growth rate stays $e^{-x}$ and amplitude scale like $\frac{1}{\alpha}$

**Right panel:** Fixed $\alpha=0.7$, varying $(c_1,c_2)$. This isolates the two independent Mittag-Leffler modes corresponding to the roots $m_1=1$ (growth) and $m_2=-2$ (decay), and shows how their linear combination reproduces the general solution structure familiar from the classical two-root ODE case.

**Verification:** At $\alpha=1$ with $c_1=c_2=1$, the series matches $e^x+e^{-2x}$ to machine precision, confirming both the boxed general solution and its correct reduction to the classical linear ODE result.

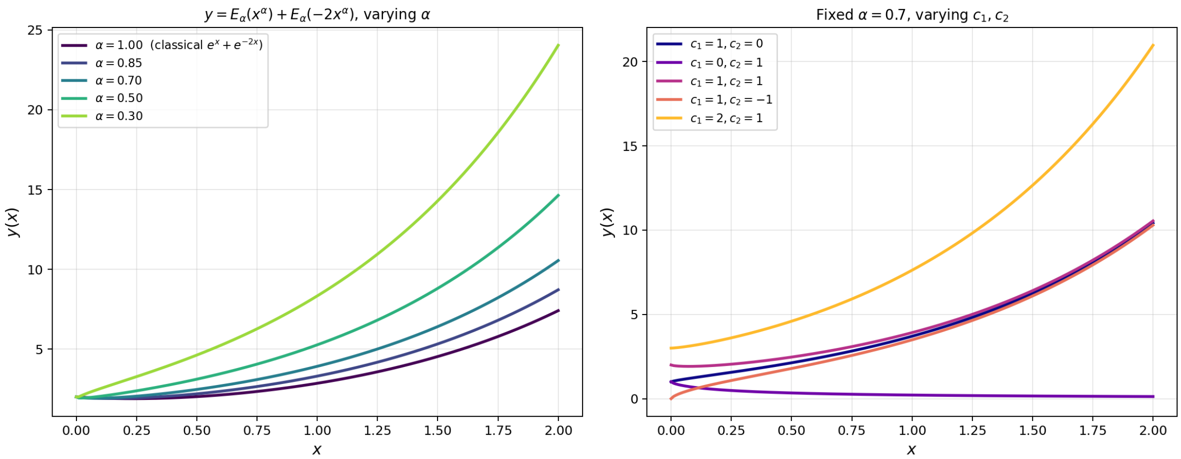


Figure 2: Solution $y(x) = c_1 E_\alpha(x^\alpha) + c_2 E_\alpha(-2x^\alpha)$ of the two-term FDE ${}_0^C D_x^{2\alpha} y + {}_0^C D_x^\alpha y - 2y = 0$ for several parameter choices, verified against the classical solution $c_1 e^x + c_2 e^{-2x}$ at $\alpha = 1$.

### 5.1.3 One-term FDE via Laplace transform

**Example 3** *Solve the given fractional differential equation by Laplace transform method*

$$ {}_0^C D_x^\alpha \, y(x) + a\, y(x) = 0, \qquad x > 0, $$

with initial condition $y(0) = C, \quad 0 < \alpha < 1$.
Applying the Laplace transform to both sides:

$$ \mathcal{L}\{{}_0^C D_x^\alpha y(x)\} + a\,\mathcal{L}\{y(x)\} = 0. $$

We have $\alpha \in (0, 1)$, so $n = 1$. Using the Laplace formula:

$$ s^\alpha Y(s) - \sum_{k=0}^{0} s^{\alpha-k-1} y^{(k)}(0) + a\,Y(s) = 0 $$
$$ Y(s) = C\frac{s^{\alpha-1}}{s^\alpha + a}. $$

Applying the Inverse Laplace Transform,

$$ \mathcal{L}\left\{x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda} \quad \Longleftrightarrow \quad \mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha - \lambda}\right\} = x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha). $$

Comparing $\frac{s^{\alpha-1}}{s^\alpha + a} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}$, therefore,$y(x) = C \cdot x^{1-1} E_{\alpha,1}(-a\,x^\alpha) = C \cdot E_{\alpha,1}(-a\,x^\alpha)$. Since $E_{\alpha,1}(\cdot) \equiv E_\alpha(\cdot)$, the solution is $\boxed{y(x) = C \cdot E_\alpha(-a\,x^\alpha)}$.
Verification with Classical Solution ($\alpha = 1$)

$$ y(x) = C \cdot E_{1,1}(-ax) = C \cdot e^{-ax}. $$

The classical ODE $y'(x) + ay(x) = 0$ with $y(0) = C$ gives:

$$ y'(x) = -a\,y(x) \quad \Longrightarrow \quad y = e^{C_1} \cdot e^{-ax}. $$

Applying $y(0) = C$: $y(x) = C\,e^{-ax}$, which matches exactly with the classical differential solution.

### 5.1.4 One-term FDE (order $\frac{1}{2}$) via LT (RL Derivative)

**Example 4** *Solve the linear homogeneous fractional differential equation:*

$$ {}_0D_x^{1/2}\, y(x) + a\, y(x) = 0, \qquad x > 0, $$

*with initial condition* $\left[{}_0D_x^{-1/2}\, y(x)\right]_{x=0} = C.$
Applying the Laplace transform to both sides of the equation:

$$\mathcal{L}\left\{{}_0D_x^{1/2}\, y(x)\right\} + a\,\mathcal{L}\{y(x)\} = 0.$$

Here $\alpha = \dfrac{1}{2}$, so $n = 1$ then using the RL Laplace transform formula

$$\begin{aligned}
&s^{1/2}Y(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_x^{1/2-k-1}\, y(x)\right]_{x=0} + a\,Y(s) = 0\\
&s^{1/2}Y(s) - s^{-1/2}\left[{}_0D_x^{-1/2}\, y(x)\right]_{x=0} + a\,Y(s) = 0\\
&s^{1/2}Y(s) - C + a\,Y(s) = 0. \quad [\text{because } \textit{Using Initial Condition}]
\end{aligned}$$

Since $Y(s) = \mathcal{L}\{y(x)\}$, we have:

$$\left(s^{1/2} + a\right)Y(s) = C \implies Y(s) = \frac{C}{s^{1/2} + a}$$

Applying the Inverse Laplace Transform,
Rewriting $Y(s)$ to match the standard formula,

$$Y(s) = \frac{C}{s^{1/2} + a} = C \cdot \frac{s^{\frac{1}{2}-1}}{s^{\frac{1}{2}} - (-a)}.$$

Comparing with $C\dfrac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}$ with $\alpha - \beta$, we identify
$\alpha = \frac{1}{2}, \qquad \alpha - \beta = \frac{1}{2} - 1 \Rightarrow \beta = \frac{1}{2}, \qquad \lambda = -a.$
Therefore, applying the inverse LT

$$\begin{aligned}
&y(x) = C \cdot x^{\beta-1}\, E_{\alpha,\beta}(\lambda x^{\alpha}) = C \cdot x^{\frac{1}{2}-1}\, E_{\frac{1}{2},\frac{1}{2}}(-a\, x^{1/2})\\
&\boxed{y(x) = C \cdot x^{-\frac{1}{2}}\, E_{\frac{1}{2},\frac{1}{2}}(-a\, x^{1/2})}.
\end{aligned}$$

#### 5.1.5 Solving a homogeneous FDE via MLF and LT method

**Example 5** *Consider the FDE*

$${}^{C}D_x^{\alpha} y(x) = \lambda y(x), \quad 0 < \alpha \le 1, \quad y(0) = 1 \quad ........(i)$$

*where* $\lambda$ *is a real constant and* ${}^{C}D_x^{\alpha}$ *denotes the CFD of order* $\alpha$ *.*

**Method 1: Direct Mittag–Leffler approach**

We propose the solution in the form of the MLF

$$y(x) = E_{\alpha}(\lambda x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^k}{\Gamma(\alpha k + 1)}.$$

Verification:

$$\begin{aligned}
{}^{C}D_x^{\alpha} E_{\alpha}(\lambda x^{\alpha}) &= {}^{C}D_x^{\alpha} \sum_{k=1}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)}\\
&= \sum_{k=1}^{\infty} \frac{\lambda^k \cdot x^{\alpha(k-1)}}{\Gamma(\alpha(k-1) + 1)}
\end{aligned}$$

Let $j = k - 1$, so when $k = 1$, $j = 0$:

$$\begin{aligned} {}^{C}D_x^{\alpha} E_{\alpha}(\lambda x^{\alpha}) &= \sum_{j=0}^{\infty} \frac{\lambda^{j+1} x^{\alpha j}}{\Gamma(\alpha j + 1)} \\ &= \lambda \sum_{j=0}^{\infty} \frac{(\lambda x^{\alpha})^j}{\Gamma(\alpha j + 1)} \\ &= \lambda\, E_{\alpha}(\lambda x^{\alpha}) \end{aligned}$$

Therefore:

$$\boxed{{}^{C}D_x^{\alpha} E_{\alpha}(\lambda x^{\alpha}) = \lambda\, E_{\alpha}(\lambda x^{\alpha})}$$

This confirms that $y(x) = E_{\alpha}(\lambda x^{\alpha})$ satisfies equation (i).

Checking Initial Condition:

$$y(0) = E_{\alpha}(0) = \frac{0^0}{\Gamma(1)} = 1$$

Therefore the solution via Mittag–Leffler approach is

$$\boxed{y(x) = E_{\alpha}(\lambda x^{\alpha})}.$$

**Method 2: Laplace transform confirmation**

Applying the Laplace transform to equation (i) [4]

$$\begin{aligned} s^{\alpha} Y(s) - s^{\alpha-1} y(0) &= \lambda Y(s) \\ s^{\alpha} Y(s) - s^{\alpha-1} &= \lambda Y(s) \\ Y(s)(s^{\alpha} - \lambda) &= s^{\alpha-1} \\ Y(s) &= \frac{s^{\alpha-1}}{s^{\alpha} - \lambda} \end{aligned}$$

Taking inverse Laplace transform using with $\beta = 1$

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha} - \lambda}\right\} \\ &= x^{1-1} E_{\alpha,1}(\lambda x^{\alpha}) \\ &= E_{\alpha}(\lambda x^{\alpha}) \end{aligned}$$

The solution is

$$\boxed{y(x) = E_{\alpha}(\lambda x^{\alpha})}$$

Comparison with Classical Solution when $\alpha = 1$, using $E_1(z) = e^z$

$$y(x) = E_1(\lambda x) = e^{\lambda x}$$

This is exactly the solution of the classical first order differential equation:

$$\frac{dy}{dx} = \lambda y(x), \quad y(0) = 1$$

Accordingly, the Mittag–Leffler function $E_{\alpha}(\lambda x^{\alpha})$ provides the fractional-order analogue of the classical exponential function $e^{\lambda x}$.

Figure 3 illustrates the behaviour of the solution $y(x) = E_{\alpha}(\lambda x^{\alpha})$ obtained via both the direct MLF approach and the Laplace transform method.

**Left panel:** $y(x)$ for fixed $\lambda = -1$, varying $\alpha$. Every curve satisfies the initial condition $y(0) = 1$, as marked on the plot. The classical exponential decay solution $e^{-x}$ is recovered in the integer-order limit

Table 1: Comparison of solution methods for Example 5.1.5

| Method | Solution | Remark |
|---|---|---|
| MLF Direct | $E_\alpha(\lambda x^\alpha)$ | Fractional solution |
| Laplace Transform | $E_\alpha(\lambda x^\alpha)$ | Confirms MLF result |
| Classical ($\alpha = 1$) | $e^{\lambda x}$ | Special case of MLF |

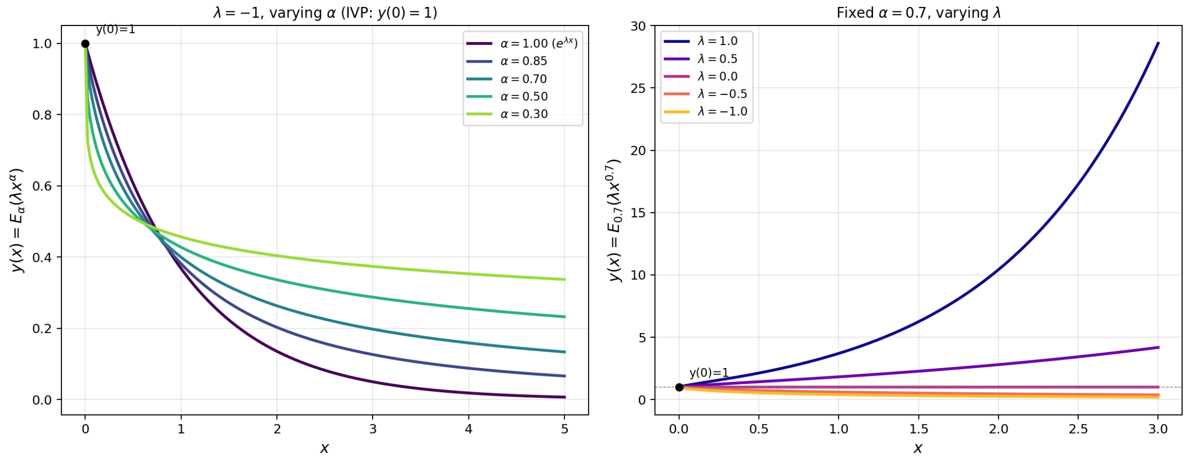


Figure 3: Solution $y(x) = E_\alpha(\lambda x^\alpha)$ of the IVP ${}^C D_x^\alpha y = \lambda y$, $y(0) = 1$, for several parameter choices, verified against the classical solution $e^{\lambda x}$ at $\alpha = 1$.

$\alpha = 1$ for smaller $\alpha$, the curve shows the characteristic fractional-relaxation signature — a sharper initial decay followed by a slow, power-law tail, in place of the pure exponential decay of the classical case.

**Right panel:** Fixed $\alpha = 0.7$, varying $\lambda$. All curves again satisfy $y(0) = 1$. Positive $\lambda$ produces Mittag-Leffler growth, $\lambda = 0$ gives the constant solution $y = 1$, and negative $\lambda$ gives decay — exactly mirroring the qualitative behaviour of the classical exponential $e^{\lambda x}$.

**Verification:** At $\alpha = 1$ with $\lambda = -1$, the series matches $e^{-x}$ to machine precision, confirming both solution methods and their correct reduction to the classical IVP result $y(x) = e^{\lambda x}$.

### 5.1.6 Solving non-homogeneous two-term FDEs involving CFD

**Example 6** *Consider the two term FDE:*

$$^C D_x^\alpha y(x) +^C D_x^\beta y(x) = h(x), \quad 0 < \beta < \alpha \leq 1, \quad y(0) = c$$

*where* $h(x)\epsilon\mathbb{C}$.
Applying Laplace transform in both sides

$$\mathcal{L}\{_0\mathcal{D}_x^\alpha y(x)\} + \mathcal{L}\{_0\mathcal{D}_x^\beta y(x)\} = \mathcal{L}\{h(x)\}$$
$$\implies \left[s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0)\right] + \left[s^\beta Y(s) - \sum_{k=0}^{n-1} s^{\beta-k-1} y^{(k)}(0)\right] = H(s)$$

$$\implies s^\alpha Y(s) - s^{\alpha-1}y(0) + s^\beta Y(s) - s^{\beta-1}y(0) = H(s)$$
$$\implies (s^\alpha + s^\beta)Y(s) - (s^{\alpha-1} + s^{\beta-1})y(0) = H(s)$$
$$\implies (s^\alpha + s^\beta)Y(s) = H(s) + (s^{\alpha-1} + s^{\beta-1}) \cdot C$$
$$\implies Y(s) = \frac{H(s)}{s^\alpha + s^\beta} + \frac{s^{\alpha-1} + s^{\beta-1}}{s^\alpha + s^\beta} \cdot C$$
$$\implies Y(s) = H(s) \cdot \frac{1}{s^\alpha + s^\beta} + \frac{s^\alpha \cdot \frac{1}{s} + s^\beta \cdot \frac{1}{s}}{s^\alpha + s^\beta} \cdot C$$

$$\implies Y(s) = H(s) \cdot \frac{1}{s^{\alpha} + s^{\beta}} + \frac{1}{s} \cdot C$$

$$\text{therefore,} y(x) = \mathcal{L}^{-1}\{G(s) \cdot H(s)\} + C \quad (ii)$$

Where we define the transfer function $G(s)$ as:

$$G(s) = \frac{1}{s^{\alpha} + s^{\beta}} = \frac{1}{s^{\alpha}\left(1 + \frac{s^{\beta}}{s^{\alpha}}\right)} = \frac{s^{-\alpha}}{s^{\beta-\alpha} + 1}$$

Thus, the corresponding kernel function $g(x)$ in the time domain is:

$$g(x) = \mathcal{L}^{-1}\left\{\frac{s^{-\alpha}}{s^{\beta-\alpha} + 1}\right\}$$

To evaluate this inverse transform, we look at the standard Mittag-Leffler Laplace pair:

$$\mathcal{L}^{-1}\left\{\frac{s^{\mu-\nu}}{s^{\mu} - \lambda}\right\} = x^{\nu-1} E_{\mu,\nu}(\lambda \cdot x^{\mu})$$

Comparing our parameters with the standard pair formulation:

$$\mu - \nu = -\beta, \qquad \mu = \alpha - \beta, \qquad \lambda = -1, \qquad \nu = \alpha$$

This explicitly yields the function $g(x)$:

$$g(x) = x^{\alpha-1} \cdot E_{\alpha-\beta,\alpha}(-1 \cdot x^{\alpha-\beta})$$

From equation (ii), let $H(s) = \mathcal{L}\{h(x)\}$ and $G(s) = \mathcal{L}\{g(x)\}$. Using the Convolution Theorem gives the final explicit integral solution for $y(x)$:

$$y(x) = \int_0^x g(x-t) h(t)\, dt + C$$
$$y(x) = \int_0^x (x-t)^{\alpha-1} \cdot E_{\alpha-\beta,\alpha}\left(-1 \cdot (x-t)^{\alpha-\beta}\right) h(t)\, dt + C.$$

#### 5.1.7 Non-homogeneous one-term FDE via LT involving CFD

**Example 7** *Solve the non-homogeneous linear fractional differential equation involving the Caputo fractional derivative:*

$${}_0^C D_x^{\alpha}\, y(x) + \lambda\, y(x) = h(x), \qquad x > 0,$$

*with initial conditions* $y^{(k)}(0) = b_k, \qquad k = 0, 1, 2, \ldots, n-1,$ *where* $n - 1 < \alpha \leq n$.
Denoting $Y(s) = \mathcal{L}\{y(x)\}$ and $H(s) = \mathcal{L}\{h(x)\}$, and substituting:

$$s^{\alpha} Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, y^{(k)}(0) + \lambda\, Y(s) = H(s). \qquad (iii)$$

Substituting $y^{(k)}(0) = b_k$:

$$\implies s^{\alpha} Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, b_k + \lambda\, Y(s) = H(s)$$

$$\implies Y(s) = \frac{H(s)}{s^{\alpha} + \lambda} + \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha} + \lambda} \cdot b_k$$

Taking $\mathcal{L}^{-1}$ of both sides:

$$y(x) = \mathcal{L}^{-1}\{H(s) \cdot G(s)\} + \sum_{k=0}^{n-1} b_k \cdot \mathcal{L}^{-1}\left\{\frac{s^{\alpha-k-1}}{s^{\alpha} + \lambda}\right\}$$

where

$$G(s) = \frac{1}{s^\alpha + \lambda} = \frac{s^0}{s^\alpha + \lambda}.$$

We have the standard inverse Laplace formula:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha - \lambda}\right\} = x^{\beta-1}\, E_{\alpha,\beta}(\lambda\, x^\alpha).$$

Comparing with $\dfrac{s^{\alpha-\beta}}{s^\alpha - \mu}$, we identify:

$$\alpha - \beta = 0 \;\Rightarrow\; \beta = \alpha, \qquad \mu = -\lambda.$$

$$\text{therefore,} g(x) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha + \lambda}\right\} = x^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda\, x^\alpha).$$

Finding the inverse of the second term for each $k = 0, 1, \ldots, n-1$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-k-1}}{s^\alpha + \lambda}\right\} = \mathcal{L}^{-1}\left\{\frac{s^{\alpha-k-1}}{s^\alpha - (-\lambda)}\right\}.$$

Comparing with $\dfrac{s^{\alpha-\beta}}{s^\alpha - \mu}$, we identify $\alpha - \beta = \alpha - k - 1$, giving $\beta = k + 1$ and $\mu = -\lambda$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-k-1}}{s^\alpha + \lambda}\right\} = x^k\, E_{\alpha,\, k+1}(-\lambda\, x^\alpha).$$

From $(iii)$, the solution is

$$y(x) = g(x) * h(x) + \sum_{k=0}^{n-1} b_k\, x^k\, E_{\alpha,\, k+1}(-\lambda\, x^\alpha).$$

Expanding the convolution

$$y(x) = \int_0^x g(x-t)\, h(t)\, dt + \sum_{k=0}^{n-1} b_k\, E_{\alpha,\, k+1}(-\lambda\, x^\alpha).$$

Substituting $g(x) = x^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda\, x^\alpha)$

$$y(x) = \int_0^x (x-t)^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda(x-t)^\alpha)\, h(t)\, dt + \sum_{k=0}^{n-1} b_k\, E_{\alpha,\, k+1}(-\lambda\, x^\alpha)$$

which, using $\mathcal{L}^{-1}\left\{\dfrac{s^{\alpha-k-1}}{s^\alpha + \lambda}\right\} = x^k\, E_{\alpha,\, k+1}(-\lambda\, x^\alpha)$, gives the complete solution:

$$\boxed{y(x) = \int_0^x (x-t)^{\alpha-1}\, E_{\alpha,\alpha}(-\lambda(x-t)^\alpha)\, h(t)\, dt + \sum_{k=0}^{n-1} b_k \cdot x^k \cdot E_{\alpha,\, k+1}(-\lambda\, x^\alpha)}$$

is the general solution.

**Classical recovery ($\alpha \to 1$).** Note that the strict domain $n-1 < \alpha < n$ excludes integer $\alpha$ for any $n$; the classical case $\alpha = 1$ is recovered as $\alpha \to 1^-$ within the $n = 1$ branch, i.e. $0 < \alpha < 1$ with the single initial condition $y(0) = b_0$. There,

$$y(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(x-t)^\alpha)\, h(t)\, dt + b_0\, E_{\alpha,1}(-\lambda x^\alpha).$$

Using $E_{1,1}(z) = e^z$, both Mittag-Leffler terms collapse as $\alpha \to 1^-$:

$$(x-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(x-t)^\alpha) \longrightarrow e^{-\lambda(x-t)},$$
$$E_{\alpha,1}(-\lambda x^\alpha) \longrightarrow e^{-\lambda x},$$

giving

$$y(x) \longrightarrow \int_0^x e^{-\lambda(x-t)}h(t)\,dt + b_0\,e^{-\lambda x},$$

which is exactly the classical variation-of-parameters solution of $y'(x) + \lambda y(x) = h(x)$, $y(0) = b_0$.

## 6 Results and discussion

Table 2: Summary of analytical solutions and their classical integer-order counterparts

| No. | FDE | Fractional Solution | Classical ($\alpha = 1$) |
|---|---|---|---|
| 5.1.1 | $\frac{d^\alpha y}{dx^\alpha} = Ay$ | $y(x) = E_\alpha(A\,x^\alpha)$ | $y(x) = e^{Ax}$ |
| 5.1.2 | $\frac{d^{2\alpha} y}{dx^{2\alpha}} + \frac{d^\alpha y}{dx^\alpha} - 2y = 0$ | $y(x) = c_1\,E_\alpha(x^\alpha) + c_2\,E_\alpha(-2x^\alpha)$ | $y(x) = C_1\,e^x + C_2\,e^{-2x}$ |
| 5.1.3 | ${}_0^C D_x^\alpha y + a\,y = 0$,<br>$y(0) = C,\ \ 0 < \alpha < 1$ | $y(x) = C \cdot E_\alpha(-a\,x^\alpha)$ | $y(x) = C\,e^{-ax}$ |
| 5.1.4 [$\alpha = \frac{1}{2}$ case] | ${}_0D_x^{1/2} y + a\,y = 0$<br>(RL derivative),<br>$\left[{}_0D_x^{-1/2} y\right]_{x=0} = C$ | $y(x) = C \cdot x^{-1/2}\,E_{\frac{1}{2},\frac{1}{2}}(-a\,x^{1/2})$ | $y(x) = C\,e^{-ax}$ |
| 5.1.5 | ${}^C D_x^\alpha y = \lambda\,y$,<br>$y(0) = 1,\ \ 0 < \alpha \le 1$ | $y(x) = E_\alpha(\lambda\,x^\alpha)$ | $y(x) = e^{\lambda x}$ |
| 5.1.6 | ${}^C D^\alpha y + {}^C D^\beta y = h(x)$,<br>$0 < \beta < \alpha \le 1$,<br>$y(0) = c$ | $y(x) = \int_0^x (x-t)^{\beta-1}$<br>$\times E_{\beta-\alpha,\,\beta}(-(x-t)^{\beta-\alpha})$<br>$\times h(t)\,dt + ce^{-x}$ | $y(x) =$<br>$\int_0^x e^{-(x-t)}h(t)\,dt + ce^{-x}$ |
| 5.1.7 | ${}_0^C D_x^\alpha y + \lambda\,y = h(x)$,<br>$y^{(k)}(0) = b_k$,<br>$n-1 < \alpha < n$ | $y(x) = \int_0^x (x-t)^{\alpha-1}$<br>$\times E_{\alpha,\alpha}(-\lambda(x-t)^\alpha)$<br>$\times h(t)\,dt$<br>$+ \sum_{k=0}^{n-1} b_k\,x^k\,E_{\alpha,\,k+1}(-\lambda\,x^\alpha)$ | $y(x) =$<br>$\int_0^x e^{-\lambda(x-t)}h(t)\,dt$<br>$+b_0\,e^{-\lambda x}$ |

Table 2 summarizes the seven examples solved in Section 5, alongside their classical integer-order counterparts at $\alpha = 1$. Looking at the results as a whole, a consistent pattern emerges: whenever the fractional derivative is Caputo (or Riemann–Liouville, where noted) and the equation is linear, the Mittag–Leffler function either alone or convolved with a forcing term is what falls out of the algebra, regardless of whether the solution route is direct series substitution or the Laplace transform.

Example 1 and Example 2 dealt with homogeneous equations solved by assuming an MLF-type ansatz and matching coefficients directly in the series. For the one-term equation ${}^C D_x^\alpha y = Ay$ (Example 1), this gave $y(x) = E_\alpha(Ax^\alpha)$ (Figure 1). For the two-term equation ${}^C D_x^{2\alpha} y + {}^C D_x^\alpha y - 2y = 0$ (Example 2), the same idea produced a genuine characteristic equation, $a^2 + a - 2 = 0$, with roots $a = 1, -2$ and general solution $y = c_1 E_\alpha(x^\alpha) + c_2 E_\alpha(-2x^\alpha)$ (Figure 2). What's worth noting is that Example 2 did not require a second, independent guess the recurrence on the series coefficients collapsed into exactly the same characteristic polynomial one would write down for the classical ODE $y'' + y' - 2y = 0$, with $e^{mx}$ simply replaced by $E_\alpha(mx^\alpha)$. Both examples reduced correctly to their classical counterparts at $\alpha = 1$, and the plotted solutions show the expected fractional relaxation behaviour: a sharper initial response followed by a heavier tail, compared to a plain exponential.

Examples 3 and 4 switch to the Laplace transform as the main solution tool rather than a series ansatz, and this is where the method starts to show its practical value. For the one-term Caputo equation ${}^C D_x^\alpha y + ay = 0$ with $y(0) = C$ (Example 3), transforming, solving algebraically for $Y(s)$, and matching against the standard MLF Laplace pair gave $y(x) = C\,E_\alpha(-ax^\alpha)$ directly, without writing out an infinite series by hand. The order-1/2 equation with the Riemann–Liouville derivative (Example 4) followed the same procedure but produced the two-parameter function $E_{1/2,1/2}(\cdot)$ rather than the one-parameter $E_\alpha$, a direct consequence of the RL initial condition being specified through a fractional integral rather than $y(0)$ itself. **Example 5** solved the IVP ${}^C D_x^\alpha y = \lambda y$, $y(0) = 1$, by both the direct MLF approach and the Laplace transform, and both routes gave the identical result $y(x) = E_\alpha(\lambda x^\alpha)$ (Figure 3). This agreement is a useful cross-check: the direct method relies on term-by-term differentiation of the MLF series, while the Laplace method does not, so the two arriving at the same answer is not a circular result.

**Examples 6 and 7** extend the framework to non-homogeneous equations, which is where the practical advantage of the Laplace approach becomes clearest. For the two-term equation ${}^C D_x^\alpha y + {}^C D_x^\beta y = h(x)$ (Example 6), transforming and inverting produced a transfer function $G(s)$ whose inverse is a shifted two-parameter Mittag–Leffler kernel $g(x) = x^{\alpha-1}E_{\alpha-\beta,\alpha}(-x^{\alpha-\beta})$; the full solution is then the convolution of this kernel with the forcing term, plus the constant contributed by the initial condition. Example 7, the general non-homogeneous one-term equation ${}^C D_x^\alpha y + \lambda y = h(x)$ with $n$ initial conditions, followed the same logic and produced the closed-form solution

$$y(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(x-t)^\alpha)\, h(t)\, dt + \sum_{k=0}^{n-1} b_k\, x^k\, E_{\alpha,k+1}(-\lambda x^\alpha).$$

This result is structurally the fractional analogue of the classical variation-of-parameters solution: the summation term is built from the same Mittag–Leffler pieces that appear in the homogeneous Examples 1–5, while the integral term is a convolution against a Mittag–Leffler kernel, mirroring how the classical solution of a linear ODE with forcing splits into a homogeneous part and a particular integral.

Taken together, the seven examples in Table 2 show that the Laplace transform is not merely an alternative route to the same MLF solutions obtained by direct substitution in Examples 1 and 2 – it is the more general of the two tools. The series ansatz works cleanly for homogeneous equations where a suitable guess is available (Examples 1, 2, 5), but it does not obviously extend to equations with an arbitrary forcing term. The Laplace method handles that case without additional guesswork (Examples 3, 4, 6, 7), at the cost of requiring the appropriate Mittag–Leffler Laplace transform pair. Every example that could be checked against a classical limit ($\alpha = 1$, or $\alpha, \beta$ integer) reduced to the expected exponential-based solution, which is the main evidence that the algebra throughout is consistent rather than merely formally plausible.

## 7 Conclusion

Our approach differs from most existing treatments of fractional differential equations in that it is purely analytical rather than numerical: exact closed-form solutions are obtained directly in terms of the Mittag–Leffler function, without discretizing the domain or approximating the fractional derivative by a finite-difference scheme. Two complementary routes were used throughout a direct Mittag–Leffler series ansatz, and the Laplace transform method and applied across several representative fractional differential equations involving the Caputo and Riemann–Liouville derivatives. The direct MLF ansatz reduces homogeneous equations to an algebraic characteristic equation in exactly the same way the classical exponential ansatz does for integer-order linear ODEs, while the Laplace transform method extends this further to equations of Riemann–Liouville type and to non-homogeneous equations with an arbitrary forcing term and multiple initial conditions, where the series ansatz alone offers no clear path forward. Where both methods were applied to the same problem, they produced identical results, confirming that the two approaches are consistent rather than independently convenient. Each method has its own advantages and limitations. The direct MLF ansatz is conceptually simple and mirrors the classical solution procedure closely, but it depends on being able to guess the correct form of the solution in advance, which becomes difficult once the equation is non-homogeneous or involves multiple fractional orders. The Laplace transform method, by contrast, is more systematic and does not require an initial guess, but it depends on having the appropriate Mittag–Leffler Laplace transform pair on hand, and the resulting solutions for non-homogeneous problems are left as convolution integrals that may not always reduce to a simple closed form for a general forcing function $h(x)$. In every case tested, however, both methods correctly reduced to the corresponding classical exponential solution at $\alpha = 1$, which provides strong support for the internal consistency of the framework as a whole.The choice between the two methods, in practice, depends on the structure of the equation at hand: the direct ansatz is preferable for simple homogeneous problems where the characteristic roots are easy to find, while the Laplace transform is the more robust choice once non-homogeneous terms, multiple derivative orders, or Riemann–Liouville-type initial conditions are involved. Extending this framework to nonlinear fractional differential equations, or to systems of coupled FDEs, would be a natural next step, since neither the series ansatz nor the Laplace transform method used here assumes linearity is present. It would also be worthwhile to investigate the convolution integrals arising in the non-homogeneous cases for specific physically motivated forcing functions $h(x)$, evaluating them either in closed form where possible or numerically otherwise, and comparing the resulting solutions against real memory-dependent systems in areas such as viscoelasticity, anomalous diffusion, and fractional-order control.

## Declaration of competing interest

The authors declare no conflict of interest.

## Data availability

No data were used.

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