Journal of Advanced College of Engineering and Management, Vol. 6, 2021

Q-ANALOGUE OF HOLDER'S AND MINKOWSKI'S INTEGRAL INEQUALITIES ON FINITE INTERVALS AND GENERALIZATION

Suresh Bhatta[1], Chet Raj Bhatta[2]

[1,2]Prof., Central Department of Mathematics, Kirtipur Campus, T.U.

Abstract

In recent years, the topic on Holder's and Minkowski's inequalities has been studied by several researchers and variety of new results has been developed on their variants, extensions and generalizations. In this paper we give the extension to the generalized q- Holder's integral inequality and by using it we also establish the generalization on q- Minkowski's integral inequality on the finite interval [a, b]

Keywords: *Holder's Inequality, Minkowski's Inequality, Jensen's Inequality, q-derivative, Jackson q- integration.*

1. Introduction and preliminaries

Classical Holder's and Minkowski's integral inequalities are defined as follows.

Let p > 1 and $\frac{1}{p}+\frac{1}{q}=1$. The discrete and integral forms of Holder's inequality are given as

$$\sum_{k=1}^{n}|a_k b_k| \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}} \dots\dots\dots (1)$$

for sequence a_i, b_i and

$$\int_a^b |f(t)g(t)|dt \le \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}} \dots\dots\dots..(2)$$

for continuous function f and g in the interval [a, b].

Let p>1. The discrete and integral forms of Minkowski's inequality are given by

$$\left(\sum_{k=1}^{n}|a_k+b_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|b_k|^p\right)^{\frac{1}{p}} \dots\dots\dots (3)$$

for sequence a_i, b_i and

$$\left(\int_a^b |f(t)+g(t)|^p\ dt\right)^{\frac{1}{p}} \le \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}} \dots\dots..(4)$$

for continuous function f and g in the interval [a, b].

The classical Holder's inequality was first found by Leonard James Rogers in 1988 and found independently by Holder (1989) and it is used to prove Minkowski's inequality. These classical inequalities have vital role in mathematical analysis, harmonic analysis, functional analysis, partial differential equations, probability and so on. By the virtue of being applicable more and more researchers have been involved for the extensions and generalization of these results not only in classical analysis but also in quantum calculus. In 2014, for the first time two researchers Jessada Tariboon and Sotiris K Ntomah [5] gave q- generalization to the holder's and Minkowski's integral inequality.

The purpose of this paper is to find the extension to the generalized Holder's inequality in the finite interval [a, b]. We also find the q- analogue of the classical Minkowski's integral inequality and generalize it in finite interval [a, b].

We present some notations and definitions from the q-calculus, which are necessary for understanding this paper. Let J:= [a, b]$\subset R$

Definition 1. [3] The q- derivative of a continuous function $f\colon J \;\to R$ at x is defined as:

$$aD_q\, f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, x\; \neq a \ldots\ldots (5)$$

For $x = a$ it is defined as $\quad aD_q f(a) = \lim_{x\to a} a\, D_q f(x)$

If $aD_q\, f(x)$ exists for all $x \in J$, then f is q- differentiable on J. Moreover, if a= 0, then (5) reduces to

$$0D_q\, f(x) = \; D_a f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

For more details, see [6].

The higher order –order q – derivatives of functions on J are also defined.

Definition 1. [3] For a continuous function $f\colon J \to R$, the second –order derivative of f on J , if $aD_q\, f$ is q- differentiable on J, denoted by $aD_q{}^2 f = aD_q(aD_q)$f. Similarly n^{th} order q- derivative $aD_q{}^n f$ can be defined on J, provided that $aD_q{}^{n-1} f$ is defined.

Definition 2. [3] Let $f\colon J \to R$be a continuous functions. Then, the q-definite integral on J is represented as

$$\int_a^x f(t)a\; d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q)a), for\; x \in J \ldots\ldots\ldots (6)$$

If a = 0 in (6), it reduces to the classical q- integral called Jackson's integral on [0, x] defined [6] as

$$\int_0^x f(t)\; 0\; d_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n), for\; x \in [0,\infty] \ldots\ldots..(7)$$

2. Auxiliary Results

Lemma (3) (**Young's Inequality**) It is one of the well- known classical inequalities and is given by for $x, y\; \geq 0, a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality

$$xy\; \leq \frac{x^a}{a} + \frac{y^b}{b} \ldots\ldots\ldots.(8)$$

is satisfied.

Lemma (4) (Jensen's Inequality) Let $f:(a,b) \to R$ be a convex function on the interval (a,b). Let $n \in N$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0,1)$ such that $\sum_{k=1}^{n} \alpha_k = 1$. Then for any $x_1, x_2, x_3, \dots \in (a,b)$ we have

$$f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leq \sum_{k=1}^{n} \alpha_k f(x_k) \dots \dots .(9)$$

Lemma (5) (Generalized Young's Inequality) : For $\boldsymbol{k = 1, 2\ 3, \dots, n}$ let $x_k \geq 0$ and $\alpha_k > 1$ such that $\sum_{k=1}^{n} \frac{1}{\alpha_k} = 1$. Then the inequality

$$\prod_{k=1}^{n} x_k \leq \sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k} \dots \dots \dots (10)$$

Proof: Without loss of generality we may take a function $f(x) = -\ln x$ defined for all $x \in (0, \infty)$. Clearly it is a convex function on its domain.

$$-ln\left(\sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k}\right) = f\left(\sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k}\right)$$

Using lemma 2.

$$\leq \sum_{k=1}^{n} \frac{1}{\alpha_k} f\left(x_k^{\alpha_k}\right)$$

$$= -\sum_{k=1}^{n} \frac{1}{\alpha_k} \ln\left(x_k^{\alpha_k}\right)$$

$$-ln\left(\sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k}\right) \leq -\sum_{k=1}^{n} \frac{1}{\alpha_k} \ln\left(x_k^{\alpha_k}\right)$$

$$\sum_{k=1}^{n} \frac{1}{\alpha_k} \ln\left(x_k^{\alpha_k}\right) \leq ln\left(\sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k}\right)$$

$$\frac{1}{\alpha_1} \ln x_1^{\alpha_1} + \frac{1}{\alpha_2} \ln x_2^{\alpha_2} + \frac{1}{\alpha_3} \ln x_3^{\alpha_3} + \dots . + \frac{1}{\alpha_n} \ln x_n^{\alpha_n} \leq \ln\left(\frac{x_1^{\alpha_1}}{\alpha_1} + \frac{x_2^{\alpha_2}}{\alpha_2} + \frac{x_3^{\alpha_3}}{\alpha_3} + \cdots \dots + \frac{x_n^{\alpha_n}}{\alpha_n}\right)$$

$$\ln x_1 + \ln x_2 + \cdots . . + \ln x_n \leq \ln\left(\frac{x_1^{\alpha_1}}{\alpha_1} + \frac{x_2^{\alpha_2}}{\alpha_2} + \frac{x_3^{\alpha_3}}{\alpha_3} + \cdots \dots + \frac{x_n^{\alpha_n}}{\alpha_n}\right)$$

$$\ln(x_1 . x_2 . x_3 \dots \dots . x_n) \leq \ln\left(\frac{x_1^{\alpha_1}}{\alpha_1} + \frac{x_2^{\alpha_2}}{\alpha_2} + \frac{x_3^{\alpha_3}}{\alpha_3} + \cdots \dots + \frac{x_n^{\alpha_n}}{\alpha_n}\right)$$

$$\therefore \prod_{k=1}^{n} x_k \leq \sum_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k}$$

Lemma (6) (Generalized Young's Inequality) : Let $x_k \geq 0$ and $\alpha_k > 0$ such that $\sum_{k=1}^{n} \alpha_k = 1$. Then the inequality

$$\prod_{k=1}^{n} x_k^{\alpha_k} \leq \sum_{k=1}^{n} \alpha_k x_k \dots \dots .(11)$$

Proof: Let $f(x) = -\ln x$ defined for all $x \in (0, \infty)$. The function f(x) is convex on its domain. So one can use lemma (2) to write.

$$-ln\left(\sum_{k=1}^{n}\alpha_k x_k\right)=f(\alpha_k x_k)$$

$$\leq -\sum_{k=1}^{n}\alpha_k f(x_k)$$

$$=-\sum_{k=1}^{n}\alpha_k \ln x_k$$

$$=-\sum_{k=1}^{n}\ln(x_k)^{\alpha_k}$$

$$=-(\ln(x_1)^{\alpha_1}+\ln(x_2)^{\alpha_2}+\ln(x_3)^{\alpha_3}+\cdots\ldots+\ln(x_n)^{\alpha_n})$$

$$=-\ln\left(x_1^{\alpha_1}.x_2^{\alpha_2}.x_3^{\alpha_3}.\ldots\ldots x_n^{\alpha_n}.\right)$$

$$=-\ln\left(\prod_{k=1}^{n}x_k^{\alpha_k}\right)$$

$$\therefore\ \ln\left(\prod_{k=1}^{n}x_k^{\alpha_k}\right)\leq\sum_{k=1}^{n}\alpha_k x_k$$

Lemma (7) [5] : For $k=1,2,3\ldots.,n$, Let $G_k\geq 0$ and $\alpha_k>1$ such that $\sum_{k=1}^{n}\frac{1}{\alpha_k}=1$. Then the inequality

$$\prod_{k=1}^{n}G_k^{\frac{1}{\alpha_k}}\leq\sum_{k=1}^{n}\frac{G_k}{\alpha_k}\ldots\ldots.(12)$$

Lemma (8) (Generalized Holder's Inequality)[5] For $i=1,2,3,\ldots.n$ and $k=1,2,3,\ldots\ldots.m$ such that the sum exists. Then the inequality

$$\sum_{i=1}^{n}\left|\prod_{k=1}^{m}G_{i,k}\right|\leq\prod_{k=1}^{m}\left(\sum_{i=1}^{n}\left|G_{i,k}\right|^{\alpha_k}\right)^{\frac{1}{\alpha_k}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(13)$$

Is valid for $\alpha_k>1$ such that $\sum_{k=1}^{m}\frac{1}{\alpha_k}=1$.

Lemma (9) (Generalized Minkowski's Inequality)[5] For $i=1,2,3,\ldots.,n$ and $k=1,2,3,\ldots.,m$ such that the sum exists. Then the inequality

$$\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{m}G_{i,k}\right|^{u}\right)^{\frac{1}{u}}\leq\sum_{k=1}^{m}\left(\sum_{i=1}^{n}\left|G_{i,k}\right|^{u}\right)^{\frac{1}{u}}\ldots\ldots\ldots\ldots\ldots\ldots.(14)$$

Is valid for $u>1$.

3. q-Holder's and q-Minkowski's inequalities on Finite Intervals

Theorem 10. [4] [q- Holder's inequality on Finite Interval]Let J= [a, b] and let $x \in J$ such that

$\frac{1}{p}+\frac{1}{q}=1$ then the inequality

$$\int_a^x |f(t)||g(t)|ad_qt \leq \left(\int_a^x |f(t)|ad_qt\right)^{\frac{1}{p}}\left(\int_a^x |g(t)|ad_qt\right)^{\frac{1}{q}} \ldots\ldots\ldots\ldots.(15)$$

Theorem 11. (q- Minkowki's Inequality on Finite Interval) Let$x \in J, 0<q<1$ and $u>1$. Let f and g be two functions such that their integral exist, then inequality

$$\left(\int_a^x |f(t)+g(t)|^u ad_qt\right)^{\frac{1}{u}} \leq \left(\int_a^x |f(t)|^u ad_qt\right)^{\frac{1}{u}}+\left(\int_a^x |g(t)|^u ad_qt\right)^{\frac{1}{u}} \ldots\ldots\ldots\ldots(16)$$

Proof:

$$\int_a^x |f(t)+g(t)|^u ad_qt = \int_a^x |f(t)+g(t)|^{u-1}|f(t)+g(t)|ad_qt$$

Using triangle inequality

$$\leq \int_a^x |f(t)||f(t)+g(t)|^{u-1}ad_qt + \int_a^x |g(t)||f(t)+g(t)|^{u-1}ad_qt \ldots\ldots..(17)$$

Let

$$I_1=(1-q)(x-a)\sum_{n=0}^{\infty}|f(q^nx+(1-q^n)a)||f(q^nx+(1-q^n)a)+g(q^nx+(1-q^n)a)|^{u-1}q^n$$

$$I_2=(1-q)(x-a)\sum_{n=0}^{\infty}|g(q^nx+(1-q^n)a)||f(q^nx+(1-q^n)a)+g(q^nx+(1-q^n)a)|^{u-1}q^n$$

Let $u_1>1$ be a number such that $\frac{1}{u}+\frac{1}{u_1}=1$

$$I_1=(1-q)(x-a)\sum_{n=0}^{\infty}\left(|f\left(q^nx+(1-q^n)a|q^{\frac{n}{u}}\right)\right)\left(|f(q^nx+(1-q^n)a) + g(q^nx+(1-q^n)a)|^{u-1}q^{\frac{n}{u_1}}\right)$$

Using Holder's inequality

$$I_1 \leq \left((1-q)(x-a)\sum_{n=0}^{\infty}|f(q^nx+(1-q^n)a)|^uq^n\right)^{\frac{1}{u}}\left((1-q)(x-a)\sum_{n=0}^{\infty}|f(q^nx+(1-q^n)a)+g(q^nx+(1-q^n)a)|^{(u-1)u_1}\right)^{1/u_1}$$

So,

$$I_1 \leq \left(\int_a^x |f(t)|^u ad_qt\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^{(u-1)u_1}ad_qt\right)^{\frac{1}{u_1}} \ldots\ldots\ldots(18)$$

Again

$$I_2 = (1-q)(x-a)\sum_{n=0}^{\infty}|g(q^n x + (1-q^n)a)||f(q^n x + (1-q^n)a) + g(q^n x + (1-q^n)a)|^{u-1}\, q^n$$

$$I_2 = (1-q)(x-a)\sum_{n=0}^{\infty}\Big(g(q^n x + (1-q^n)a)\Big)q^{\frac{n}{u}}\left(|f(q^n x + (1-q^n)a) + g(q^n x + (1-q^n)a)|^{u-1} q^{\frac{n}{u_1}}\right)$$

Using Holder's inequality

$$I_2 \le \left((1-q)(x-a)\sum_{n=0}^{\infty}|g(q^n x + (1-q^n)a)|^u q^n\right)^{\frac{1}{u}}\left((1-q)(x-a)\sum_{n=0}^{\infty}|f(q^n x + (1-q^n)a) + g(q^n x + (1-q^n)a)|^{(u-1)u_1}\right)^{1/u_1}$$

$$I_2 \le \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^{(u-1)u_1}\, {}_a d_q t\right)^{\frac{1}{u_1}} \ldots\ldots\ldots (19)$$

From (17), (18) and (19) we get

$$\int_a^x |f(t)+g(t)|^u\, {}_a d_q t \le I_1 + I_2$$

$$\int_a^x |f(t)+g(t)|^u\, {}_a d_q t \le \left(\int_a^x |f(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^{(u-1)u_1}\, {}_a d_q t\right)^{\frac{1}{u_1}} + \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^{(u-1)u_1}\, {}_a d_q t\right)^{\frac{1}{u_1}}$$

$$\int_a^x |f(t)+g(t)|^u\, {}_a d_q t \le \left(\int_a^x |f(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u_1}} + \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}\left(\int_a^x |f(t)+g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u_1}}$$

$$\frac{\int_a^x |f(t)+g(t)|^u\, {}_a d_q t}{\left(\int_a^x |f(t)+g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u_1}}} \le \left(\int_a^x |f(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}} + \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}$$

$$\left(\int_a^x |f(t)+g(t)|^u\, {}_a d_q t\right)^{1-\frac{1}{u_1}} \le \left(\int_a^x |f(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}} + \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}$$

$$\left(\int_a^x |f(t)+g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}} \le \left(\int_a^x |f(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}} + \left(\int_a^x |g(t)|^u\, {}_a d_q t\right)^{\frac{1}{u}}$$

This completes the proof.

5. Main Results

Theorem 12 (Generalized q –Holder's Inequality on Finite Interval). Let $J = [a, b]$ and

let $x \in J$, $0 < q < 1$ and $p_i > 1$ such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$.Let $f_1, f_2, f_3, \dots .. f_n$ be functions such that their integrals exists. Then the inequality

$$\int_a^x \left| \prod_{i=1}^{n} f_i(t) \right| a d_q t \le \prod_{i=1}^{n} \left(|f_i(t)|^{p_i} \, a d_q t \right)^{\frac{1}{p_i}} \dots \dots \dots (20)$$

holds.

Proof:

$$\int_a^x \left| \prod_{i=1}^{n} f_i(t) \right| a d_q t \le (1-q)(x-a) \sum_{k=1}^{\infty} \prod_{i=1}^{n} \left| f(q^k x + (1-q^k)a) \right| q^k$$

Using lemma (8) this can be written as

$$\le (1-q)(x-a) \prod_{i=1}^{n} \left(\sum_{k=1}^{\infty} \left| f(q^k x + (1-q^k)a \right|^{p_i} q^k \right)^{\frac{1}{p_i}}$$

$$= \left[(1-q)(x-a)\right]^{\sum_{i=1}^{n} \frac{1}{p_i}} \prod_{i=1}^{n} \left(\sum_{k=1}^{\infty} \left| f(q^k x + (1-q^k)a) \right|^{p_i} q^k \right)^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{n} \left[(1-q)(x-a)\right]^{\frac{1}{p_i}} \prod_{i=1}^{n} \left(\sum_{k=1}^{\infty} \left| f(q^k x + (1-q^k)a) \right|^{p_i} q^k \right)^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{n} \left((1-q)(x-a) \sum_{k=1}^{\infty} \left| f(q^k x + (1-q^k)a \right|^{p_i} q^k \right)^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{n} \left(\int_a^x |f_i(t)|^{p_i} \, a d_q t \right)^{\frac{1}{p_i}}$$

$$\int_a^x \left| \prod_{i=1}^{n} f_i(t) \right| a d_q t \le \prod_{i=1}^{n} \left(\int_a^x |f_i(t)|^{p_i} \, a d_q t \right)^{\frac{1}{p_i}}$$

This completes the proof.

Theorem 13(Generalized q- Minkowski's Inequality on Finite Interval). Let $J = [a, b]$ and let $x \in J, 0 < q < 1$ and $s > 1$. Let $f_1, f_2, \dots \dots, f_n$ be the functions such that integral exists. Then the inequality

$$\left(\int_a^x \left| \sum_{i=1}^{n} f_i(t) \right|^s a d_q t \right)^{\frac{1}{s}} \le \sum_{i=1}^{n} \left(\int_a^x |f_i(t)|^s \, a d_q t \right)^{\frac{1}{s}} \dots \dots \dots (21)$$

holds true.

Proof: We prove the theorem by using the principle of mathematical induction on n. For n= 2 the inequality reduces to the theorem (11). So assume that the theorem is true for n= k, that is

$$\left(\int_a^x \left|\sum_{i=1}^{k} f_i(t)\right|^s ad_qt\right)^{\frac{1}{s}} \le \sum_{i=1}^{k}\left(\int_a^x |f_i(t)|^s\, ad_qt\right)^{\frac{1}{s}}$$

To complete the proof of the theorem it suffices to show that whenever it is true for n= k , it is also true for n = k+1.

$$\left(\int_a^x \left|\sum_{i=1}^{k+1} f_i(t)\right|^s ad_qt\right)^{\frac{1}{s}} \le \left(\int_a^x \left|\sum_{i=1}^{k} f_i(t) + f_{k+1}(t)\right|^s ad_qt\right)^{\frac{1}{s}}$$

$$\le \left(\int_a^x \left|\sum_{i=1}^{k} f_i(t)\right|^s ad_qt\right)^{\frac{1}{s}} + \left(\int_a^x |f_{k+1}|^s ad_qt\right)^{\frac{1}{s}}$$

$$\le \sum_{i=1}^{k}\left(\int_a^x |f_i(t)|^s\right)^{\frac{1}{s}} + \left(\int_a^x |f_{k+1}|^s ad_qt\right)^{\frac{1}{s}}$$

$$= \sum_{i=1}^{k+1}\left(\int_a^x |f_i(t)|^s\, ad_qt\right)^{\frac{1}{s}}$$

$$\left(\int_a^x \left|\sum_{i=1}^{k+1} f_i(t)\right|^s ad_qt\right)^{\frac{1}{s}} \le \sum_{i=1}^{k+1}\left(\int_a^x |f_i(t)|^s\, ad_qt\right)^{\frac{1}{s}}$$

So, by the induction hypothesis it is true for all n .

6. Conclusion

In this paper, we provided the simple proof of AM-GM type inequality and generalized Young's inequality. We investigate the q-Holder's integral inequality and we proved generalized q-Holder's integral inequality in the finite interval [a, b]. We also proved q- analogue of Minkowski's integral inequality for two functions and generalized it on finite interval [a,b].

References

1) Bashir Ahamad, Sotiris Nytouyas, H.JessadaTariboon(2016), Quantum Calculus,*"new concepts, impulsive IVPs and BVPsinequalities"*Trends in Abstract and Applied Analysis Vol(4), World Scientific Publishing Co. Pt .Ltd., ISSN: 2424-8746

2) F.H. Jacson., *"On a q-definite integrals"*, Quarterly Journal of Pure and Applied Mathematics, P.P 193-203(1910).

3) JessadaTariboon and Sotiris K Ntouyas,*"Quantum calculus on finite intervals and applications to impulsive difference equations",*Advances in Differential equations 2013:282(2013)

4) JessadaTariboon and Sotiris K Ntouyas (2014)., *"Quantum integral inequalities on finite intervals,"*,Journal of Inequalities and Application 2014:212(2014)

5) KwaraNantomah,*"Generalized Holder's and Minkowski's Inequalities for Jackson's q-integral and some application to the Incomplete q-Gamma Functions"*,Abstract and Applied Analysis, Volume 2017, Article ID 9796873, 6 pages(2017).

6) Victor Kac, Poknan Cheung, *"Quantum Calculus",*Springer, North America (2001)