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Singular Factorization of an Arbitrary Matrix

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Abstract: In this paper, we study the Singular Value Decomposition of an arbitrary matrix A_{nxm}, especially its subspaces of activation, which leads in natural manner to the pseudo inverse of Moore -Bjenhammar - Penrose. Besides, we analyze the compatibility of linear systems and the uniqueness of the corresponding solution and our approach gives the Lanczos classification for these systems.

Keywords: SVD, Compatibility of linear systems, Pseudo inverse of a matrix

1. Introduction

For any real matrix A_{nxm}, Lanczos [18] constructs the matrix:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \tag{1}$$

and he studies the eigenvalue problem:

$$S\vec{\omega} = \lambda\vec{\omega}, \tag{2}$$

where the proper values are real because S is a real symmetric matrix. Besides,

$$\text{rank } A \equiv p = \text{Number of positive eigenvalues of } S, \tag{3}$$

such that $1 \leq p \leq \min(n, m)$. Then the singular values or canonical multipliers, thus called by Picard [26] and Sylvester [31], respectively, follow the scheme:

$$\lambda_1, \lambda_2, \dots, \lambda_p, -\lambda_1, -\lambda_2, \dots, -\lambda_p, 0, 0, \dots, 0, \tag{4}$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of S, named 'essential axes' by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \begin{matrix} n \\ m \end{matrix}, \qquad (5)$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{nxm}\vec{v}_{mx1} = \lambda\,\vec{u}_{nx1}\,, \qquad A^T{}_{mxn}\vec{u}_{nx1} = \lambda\,\vec{v}_{mx1}\,, \qquad (6)$$

hence

$$A^T A\vec{v} = \lambda^2\vec{v}\,, \qquad AA^T\vec{u} = \lambda^2\vec{u}\,, \qquad (7)$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = \left(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\right), \qquad V_{mxp} = \left(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\right), \qquad (8)$$

verifying $U^T U = V^T V = I_{pxp}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk}\,, \qquad (9)$$

therefore $\vec{\omega}_j \cdot \vec{\omega}_k = 2\delta_{jk},\ j,k = 1,2,\dots,p$. Thus, the Singular Value Decomposition (SVD) express that *A* is the product of three matrices [18 - 21]:

$$A_{nxm} = U_{nxp}\Lambda_{pxp}V^T{}_{pxm}\,, \qquad \Lambda = \mathrm{Diag}\left(\lambda_1, \lambda_2, \dots, \lambda_p\right). \qquad (10)$$

This relation tells that in the construction of *A* we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to *A*. This approach of Lanczos is similar to the methods in [15, 16, 27, 28]. It can be considered that Jordan [15, 16], Sylvester [30, 31] and Beltrami [2] are the founders of the SVD [29], and there is abundant literature [4, 6, 7, 11, 30, 34] on this matrix factorization and its applications.

The rest of the paper is planned as follows: In Section 2, we realize an analysis of the proper vectors $\vec{\omega}_j, j = 1, \dots, n+m$, associated to the eigenvalues (4), which leads to the subspaces of activation of *A* with the pseudo inverse of Moore [22], Bjerhammar [3] and Penrose [25]. In Section 3, we study the compatibility of linear systems, with special emphasis in the important participation of the null singular value and its corresponding eigenvectors. Finally, Section 4 concludes the paper.

2. Subspaces of Activation and Natural Inverse Matrix

From (6), the proper vectors associated with the positive eigenvalues verify:

$$A\vec{v}_j = \lambda_j\vec{u}_j\,, \qquad A^T\vec{u}_j = \lambda_j\vec{v}_j\,, \qquad j = 1, \dots, p \qquad (11)$$

then

$$A(-\vec{v}_j) = (-\lambda_j)\,\vec{u}_j\,, \qquad A^T\vec{u}_j = (-\lambda_j)(-\vec{v}_j), \tag{12}$$

that is,

$$S\begin{pmatrix}\vec{u}_k\\ \vec{v}_k\end{pmatrix} = \lambda_k\begin{pmatrix}\vec{u}_k\\ \vec{v}_k\end{pmatrix} \quad \text{implies} \quad S\begin{pmatrix}\vec{u}_k\\ -\vec{v}_k\end{pmatrix} = (-\lambda_k)\begin{pmatrix}\vec{u}_k\\ -\vec{v}_k\end{pmatrix}, \tag{13}$$

therefore, the eigenvectors $\begin{pmatrix}\vec{u}_j\\ \vec{v}_j\end{pmatrix}$ and $\begin{pmatrix}\vec{u}_j\\ -\vec{v}_j\end{pmatrix}$ correspond to the proper values $\lambda_1, \dots, \lambda_p$ and $-\lambda_1, \dots, -\lambda_p,$ respectively. Thus we must have $n + m - 2p$ eigenvectors connected to $\lambda = 0$, which is denoted by $\vec{\omega}_r^{(0)}$, and from (6) we further have:

$$\vec{\omega}_j^{(0)} = \begin{pmatrix}\vec{u}_j^{(0)}\\ 0\\ \vdots\\ 0\end{pmatrix}\begin{matrix}n\\ \\ m\end{matrix}\,, \qquad A^T\vec{u}_j^{(0)} = \vec{0}\,, \quad j = 1, \dots, n-p, \tag{14}$$

$$\vec{\omega}_{(n-p)+k}^{(0)} = \begin{pmatrix}0\\ \vdots\\ 0\\ \vec{v}_k^{(0)}\end{pmatrix}\begin{matrix}n\\ \\ m\end{matrix}\,, \qquad A\vec{v}_k^{(0)} = \vec{0}\,, \quad k = 1, \dots, m-p. \tag{15}$$

The conditions (14) and (15) can be multiplied by A and A^T, then $\vec{u}_j^{(0)}$ and $\vec{v}_k^{(0)}$ are eigenvectors of the Gram matrices AA^T and A^TA:

$$(AA^T)_{nxn}\,\vec{u}_j^{(0)} = \vec{0}\,, \qquad (A^TA)_{mxm}\,\vec{v}_k^{(0)} = \vec{0} \tag{16}$$

but by (7) these matrices have p proper vectors for $\lambda_1, \dots, \lambda_p$, therefore only there are $n - p$ and $m - p$ vectors $\vec{u}_j^{(0)}$ and $\vec{v}_k^{(0)}$, that can be selected with orthonormality:

$$\vec{u}_j^{(0)} \cdot \vec{u}_r^{(0)} = \delta_{jr}\,, \qquad \vec{v}_k^{(0)} \cdot \vec{v}_q^{(0)} = \delta_{kq} \tag{17}$$

that is, $\vec{\omega}_j^{(0)} \cdot \vec{\omega}_k^{(0)} = \delta_{jk}$, then $\{\vec{u}_j^{(0)}\}$ and $\{\vec{v}_k^{(0)}\}$ are bases for the Kernel A^T and Kernel A, respectively.

If we employ (10) in (14), SVD of A results $V\Lambda U^T\vec{u}_j^{(0)} = \vec{0}$, whose multiplication by the left with $\Lambda^{-1}V^T$ [remembering that $U^TU = V^TV = I$], gives the compatibility condition:

$$U^T\vec{u}_j^{(0)} = \vec{0} \quad \Rightarrow \quad \vec{u}_r \cdot \vec{u}_j^{(0)} = 0, \quad r = 1, \dots, p\ ; \quad j = 1, \dots, n-p, \tag{18}$$

equivalently

$$\text{Col}\,U \perp \vec{u}_k^{(0)}\,, \quad k = 1, \dots, n-p. \tag{19}$$

Similarly, if we use SVD into (15) and we multiply by $\Lambda^{-1}U^T$:

$$V^T\vec{v}_k^{(0)} = \vec{0}\,, \qquad \vec{v}_r \cdot \vec{v}_k^{(0)} = 0, \quad r = 1, \dots, p\,; \quad k = 1, \dots m - p \tag{20}$$

$$\therefore \quad \text{Col } V \;\perp\; \vec{v}_j^{(0)}\,, \quad j = 1, \dots, m - p. \tag{21}$$

It is convenient to make two remarks:

Remark 1: From $A = U\Lambda V^T$ is evident that the matrices U, Λ and V permit to construct A, but is useful to know more about the structure of A and its transpose:

$$A = (\vec{a}_1 \dots \vec{a}_m), \qquad A^T = (\vec{c}_1 \dots \vec{c}_n), \tag{22}$$

where $(\vec{a}_j)_{nx1}$ and $(\vec{c}_k)_{mx1}$ are the corresponding columns. Then from (10) we obtain the expressions:

$$\vec{a}_j = \lambda_1 v_1^{(j)}\vec{u}_1 + \cdots + \lambda_p v_p^{(j)}\vec{u}_p, \;\; j = 1, \dots m, \qquad \vec{c}_k = \lambda_1 u_1^{(k)}\vec{v}_1 + \cdots + \lambda_p u_p^{(k)}\vec{v}_p, \;\; k = 1, \dots, n \tag{23}$$

with the notation:

$$v_r^{(j)} = j \text{ th} - \text{component of } \vec{v}_r\,, \tag{24}$$

and similar for $u_r^{(k)}$; we observe that $\vec{c}_k^T$ are the rows of A.

From (23) are immediate the equalities of subspaces:

$$\text{Col } A = \text{Col } U\,, \qquad \text{Row } A = \text{Col } V\,, \tag{25}$$

but dim Col $U =$ dim Col $V = p$, then:

$$\text{rank } A = \text{dim Col } A = \text{dim Row } A = p \tag{26}$$

in according with (3).

Remark 2: We have the rank-nullity theorem [24, 32, 33]:

$$\text{dim (Kernel } A) + \text{rank } A = m, \tag{27}$$

therefore dim (Kernel A) $= m - p$, by this reason there are $(m - p)$ vectors $\vec{v}_k^{(0)}$ with the property (15). Besides,

$$\text{dim (Kernel } A^T) + \text{rank } A^T = n, \tag{28}$$

but rank $A^T =$ rank $A = p$, then dim (Kernel A^T) $= n - p$ in harmony with the $(n - p)$ vectors $\vec{u}_j^{(0)}$ verifying (14).

If A_{nxm} acts on an arbitrary vector $\vec{x}\; \varepsilon\; E_m$ produces a vector $\vec{y}\; \varepsilon\; E_n$, with the decompositions:

$$\vec{x} = \vec{x}^{(0)} + \vec{x}_{\mathrm{CV}}\,, \qquad \vec{y} = \vec{y}^{(0)} + \vec{y}_{\mathrm{CU}}\,, \tag{29}$$

where

$$\vec{x}^{(0)}\ \varepsilon\ \text{Kernel}\ A\,,\quad \vec{x}_{\mathrm{CV}}\ \varepsilon\ \text{Col}\ V, \qquad A\vec{x}^{(0)} = \vec{0}, \quad \vec{x}^{(0)} \cdot \vec{x}_{\mathrm{CV}} = 0, \tag{30}$$

$$\vec{y}^{(0)}\ \varepsilon\ \text{Kernel}\ A^T,\ \ \vec{y}_{\mathrm{CU}}\ \varepsilon\ \text{Col}\ U, \qquad A^T\vec{y}^{(0)} = \vec{0}, \quad \vec{y}^{(0)} \cdot \vec{y}_{\mathrm{CU}} = 0,$$

thus we say that A is activated into the subspaces Col U and Col V.

Therefore, $A\vec{x} = A\vec{x}_{\mathrm{CV}} = \vec{y}$ and in the construction of $\vec{y}$ we lost the information about $\vec{x}^{(0)}$, then it is not possible to recover $\vec{x}$ from $\vec{y}$, that is, it is utopian to search for an 'inverse matrix' acting on $\vec{y}$ to give $\vec{x}$. However, when $\vec{x}^{(0)} = \vec{0}$ and $\vec{y}^{(0)} = \vec{0}$ we can introduce a 'natural inverse matrix', thus named it by Lanczos, which coincides with the pseudo inverse of Moore [22], Bjerhammar [3] and Penrose [25]:

"Any matrix A_{nxm}, restricted to its subspaces of activation, always can be inverted". (31)

In fact, if $\vec{x}\ \varepsilon$ Col V is an arbitrary vector, $\vec{x} = q_1\vec{v}_1 + \cdots + q_p\vec{v}_p$, then from (6):

$$A\vec{x} = \lambda_1 q_1 \vec{u}_1 + \cdots + \lambda_p q_p \vec{u}_p = \vec{y}\ \ \varepsilon\ \text{Col}\ U, \tag{32}$$

and now we search the inverse natural $A_N^{-1}{}_{mxn}$ such that:

$$A_N^{-1}\,\vec{y} = \vec{x}\,, \tag{33}$$

or more general:

$$A_N^{-1}A\vec{x} = \vec{x}, \quad \forall\ \vec{x}\ \varepsilon\ \text{Col}\ V, \qquad AA_N^{-1}\vec{y} = \vec{y}, \quad \forall\ \vec{y}\ \varepsilon\ \text{Col}\ U. \tag{34}$$

If the decomposition (10) is applied to (32), we deduce the natural inverse matrix:

$$A_N^{-1}{}_{mxn} = V_{mxp}\,\Lambda_{pxp}^{-1}\,U_{pxn}^T, \tag{35}$$

satisfying (33) and (34). With (35), it is easy to prove the properties [24, 32]:

$$AA_N^{-1}A = A, \qquad A_N^{-1}AA_N^{-1} = A_N^{-1}, \qquad (AA_N^{-1})^T = AA_N^{-1}, \qquad (A_N^{-1}A)^T = A_N^{-1}A, \tag{36}$$

which characterize the pseudo inverse of Moore - Bjerhammar - Penrose, that is, the inverse matrix [8, 9, 12] of these authors coincides with the natural inverse (35) deduced by Lanczos [18 - 21].

In the SVD only participate the positive proper values of *S*, without the explicit presence of the vectors $\vec{u}_j^{(0)}$ and $\vec{v}_k^{(0)}$ associated with the null eigenvalue, then it is natural to investigate the role performed by the information related with $\lambda = 0$. In Section 3, we study linear systems where A is the corresponding matrix of coefficients, and we exhibit that the $\vec{u}_j^{(0)}$ permit to analyze the compatibility of such systems; besides, when they are compatibles then with the $\vec{v}_k^{(0)}$, we search if the solution is unique. In other words, the null eigenvalue does not participates when we consider to A as an algebraic operator and we construct its factorization (10), but $\lambda = 0$ is important if A acts as the matrix of coefficients of a linear system.

3. Compatibility of Linear Systems

A Linear System of n equations with m unknowns can be written in the matrix form:

$$A_{nxm}\,\vec{x}_{mx1} = \vec{b}_{nx1}\,, \tag{37}$$

where (10) implies that $U\Lambda V^T\vec{x} = \vec{b}$ whose multiplication by $\vec{u}_j^{(0)\,T}$ gives the compatibility conditions:

$$\vec{u}_j^{(0)} \cdot \vec{b} = 0, \qquad j = 1, \dots, n-p \tag{38}$$

due to (19). Then the system (37) is compatible if $\vec{b}$ is orthogonal to all independent solutions of the adjoint system $A^T\vec{u} = \vec{0}$, therefore:

$$\text{"}A\vec{x} = \vec{b} \quad \text{has solution if} \quad \vec{b}\ \varepsilon\ \text{Col}\ U\text{"}, \tag{39}$$

which is the traditional formulation [6] of the compatibility condition for a given linear system. From (25) and (39) is clear that A and the augmented matrix $(A\,\vec{b})$ have the same column space:

$$\text{Col}\ A = \text{Col}\ \left(A\,\vec{b}\right) = \text{Col}\ U, \tag{40}$$

thus at the books [32] we find the result:

$$\text{"}A\vec{x} = \vec{b} \quad \text{is compatible if} \quad \text{rank}\ A = \text{rank}\ (A\,\vec{b})\text{"}. \tag{41}$$

If $\vec{b}\ \varepsilon\ \text{Col}\ U$, then from (11):

$$\vec{b} = b^{(1)}\vec{u}_1 + \dots + b^{(p)}\vec{u}_p = A\vec{Q}\,, \qquad \vec{Q} = \frac{b^{(1)}}{\lambda_1}\vec{v}_1 + \dots + \frac{b^{(p)}}{\lambda_p}\vec{v}_p, \tag{42}$$

and (37) leads to:

$$A\left(\vec{x} - \vec{Q}\right) = \vec{0}. \tag{43}$$

The set of solutions of (43) is the Kernel A with dimension $(m-p)$ due to (27), therefore (43) has the unique solution $\vec{x} - \vec{Q} = \vec{0}$ when $p = m$, that is, when rank A coincides with the number of unknowns we have not vectors $\vec{v}_k^{(0)} \neq \vec{0}$ verifying $A\vec{v}_k^{(0)} = \vec{0}$. Then:

"The compatible system $A\vec{x} = \vec{b}$ has unique solution only when $p = m$", (44)

besides from (24) and (42) we obtain that $b^{(k)} = \vec{b} \cdot \vec{u}_k$, $\vec{x} = \vec{Q}$ and:

$$x_r = Q^{(r)} = \frac{b^{(1)}}{\lambda_1} v_1^{(r)} + \cdots + \frac{b^{(p)}}{\lambda_p} v_p^{(r)} = \vec{b} \cdot \vec{t}_r \,, \quad r = 1, \ldots, m \tag{45}$$

where

$$\vec{t}_r = \frac{v_1^{(r)}}{\lambda_1} \vec{u}_1 + \cdots + \frac{v_p^{(r)}}{\lambda_p} \vec{u}_p \ \ \varepsilon \text{ Col } U, \tag{46}$$

thus the value of each unknown is the projection of $\vec{b}$ onto each vector (46). In consequence, $\vec{b}\ \varepsilon \text{ Col } U$ guarantees the solution of (37), and it is unique only if $p = m$.

Besides, from (42) we see that the solution $\vec{x} = \vec{Q}$ implies that $\vec{x}\ \varepsilon \text{ Col } V$, then we have the system $A\vec{x} = \vec{b}$ where $\vec{x}$ and $\vec{b}$ are totally embedded into Col V and Col U, respectively, that is, $\vec{x}$ and $\vec{b}$ are into the subspaces of activation of A, thus from (32) and (33) there is the natural inverse A_N^{-1} such that:

$$\vec{x} = A_N^{-1} \vec{b} = V_{mxm} \Lambda_{mxm}^{-1} U_{mxn}^T \vec{b} = V\Lambda^{-1} \begin{pmatrix} b^{(1)} \\ \vdots \\ b^{(m)} \end{pmatrix} = V \begin{pmatrix} \frac{b^{(1)}}{\lambda_1} \\ \vdots \\ \frac{b^{(m)}}{\lambda_m} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{b^{(1)}}{\lambda_1} v_1^{(1)} + \cdots + \frac{b^{(m)}}{\lambda_m} v_m^{(1)} \\ \vdots \\ \frac{b^{(1)}}{\lambda_1} v_1^{(m)} + \cdots + \frac{b^{(m)}}{\lambda_m} v_m^{(m)} \end{pmatrix}, \quad p = m, \tag{47}$$

in according with (45). The vectors (46) are important because their inner products with $\vec{b}$ give the solution of (37) via (45), and they also are remarkable because permit to construct the natural inverse:

$$A_N^{-1}{}_{mxn} = (\vec{t}_1\, \vec{t}_2 \ldots\, \vec{t}_m)^T, \quad p = m. \tag{48}$$

Lanczos [6] considers three situations:

i) $n < m$: The linear system is under-determined because it has more unknowns than equations, and from $1 \le p \le \min(n, m)$ is impossible the case $p = m$, therefore, if (37) is compatible then its solution cannot be unique.

ii) $n = m$: The system is even-determined with unique solution when $p = m$, that is, if $\det A \neq 0$. In this case also $p = n$, we have not vectors $\vec{u}_j^{(0)} \neq \vec{0}$, thus $\vec{b}\ \varepsilon \text{ Col } U$ and automatically the system is compatible.

iii) $n > m$: The linear system is over-determined, and by $1 \le p \le \min(n, m)$ can occur the case $p = m$ for unique solution if the system is compatible.

Hence it is immediate the classification of linear systems introduced by Lanczos [21]:

Free and complete: $p = n = m$, unique solution,

Restricted and complete:

$p = m < n,$ over-determined, unique solution, (49)

Free and incomplete: $p = n < m,$ under-determined, non-unique solution,

Restricted and incomplete: $p < n$ and $p < m,$ solution without uniqueness,

with the meaning:

Free: The conditions (30) are satisfied trivially.

Restricted: It is necessary to verify that $\vec{b}\ \varepsilon$ Col U. (50)

Complete: The solution has uniqueness.

Incomplete: Non-unique solution.

When $p \neq m,$ the homogeneous system $A\vec{v} = \vec{0}$ has the non-trivial solutions $\vec{v}_k^{(0)}$, then from (27) we conclude that the general solution of (37) is:

$$\vec{x} = \vec{Q} + c_1\vec{v}_1^{(0)} + \cdots + c_{m-p}\vec{v}_{m-p}^{(0)}, \tag{51}$$

where the c_k are arbitrary constants.

4. Conclusion

With the SVD we can find the subspaces of activation of A, and it leads to the natural inverse [6, 26-28] of any matrix, known it in the literature as the Moore-Penrose pseudo inverse. Besides, the SVD gives a better understanding of the compatibility of linear systems. On the other hand, Lanczos [21] showed that the Singular Value Decomposition provides a universal platform to study linear differential and integral operators for arbitrary boundary conditions. We note that the term 'singular value' was introduced by Green [10] (see [5] too) in his studies on electromagnetism. The SVD is very useful to study the rotation matrix in classical mechanics [14] and to comprehend the matrix technique to deduce gauge transformations of Lagrangians [17]. For a graphic example of the use of the SVD in image processing, we refer see [1]; and for its use in cryptography, we refer [23]. Heat [13] mentions software for singular value computations.

References

[1] Andrews HC and Patterson CL (1975), Outer product expansions and their uses in digital image processing. *Am. Math. Monthly,* **82**: 1-13.

[2] Beltrami E (1873), Sulle funzioni bilineari. *Giornale di Mathematische,* **11:** 98-106.

[3] Bjerhammar A (1951), Application of calculus of matrices to method of least squares, with special references to geodetic calculations. *Trans. Roy. Inst. Tech. Stockholm*, (49): 1-86.

[4] Blank SJ, Krikorian N and Spring D (1989), A geometrically inspired proof of the singular value decomposition. *Am. Math. Monthly* **96**(3): 238-239.

[5] Cannell DM (2001), George Green, mathematician & physicist *1793-1841. SIAM, Philadelphia*, USA.

[6] Eckart C and Young G (1939), A principal axis transformation for non-hermitian matrices. *Bull. Amer. Math. Soc.,* **45**(2) : 118-121.

[7] Gaftoi V, López-Bonilla J and Ovando G (2007), *Singular value decomposition and Lanczos potential*, in "Current topics in quantum field theory research", Ed. O. Kovras, Nova Science Pub., New York, Chap. 10, 313-316.

[8] Golub GH and Kahan W (1965), Calculating the singular values and pseudo inverse of a matrix. *SIAM J. Numer.Anal.* B**2**: 205-224.

[9] Greville TNE (1960), Some applications of the pseudo inverse of a matrix. *SIAM Rev.,* **2 (1) :** 15-22.

[10] Green G (1828), An essay on the application of mathematical analysis to the theories of electricity and magnetism. Private subscription, only fifty copies, reprinted in three parts at *J. Reine Angewand.Math.* **39** (1850) 73-89, **44** (1852) 356-374 and **47** (1854): 161-212.

[11] Guerrero IM, López-Bonilla J and Rosales RL (2012), SVD applied to Dirac super matrix, *The SciTech, J. Sci. & Tech.* Special Issue, Aug. 111-114.

[12] Hazra AK (2006), *Matrix algebra, calculus and generalized inverse.* Cambridge Int. Sci. Pub.

[13] Heat MT (1997), *Scientific computing: An introductory survey.* Chap. 4, McGraw-Hill, New York.

[14] Hernández GA, López-Bonilla J and Rivera RJ (2012), A similarity transformation for the rotation matrix. *Int. J. Math. Eng. Sci*., **1(1):** 8-12.

[15] Jordan C (1874), Mémoire sur les forms bilinéaires. *J. de Mathématiques Pures et Appliquées*, Deuxieme Série, **19**: 35-54.

[16] Jordan C (1874), Sur la réduction des formes bilinéaires. *Compt. Rend. Acad. Sci. Paris*, **78**: 614 - 617.

[17] Lam P, López-Bonilla J, López-Vázquez R and Ovando G (2015), Matrix method to construct point symmetries of Lagrangians, *Bull. of Kerala Mathematics Association,* **12**(1): 43-52.

[18] Lanczos C (1958), Linear systems in self-adjoint form. *Am. Math. Monthly,* **65**(9): 665 - 679.

[19] Lanczos C (1960), Extended boundary value problems. *Proc. Int. Congr. Math.* Edinburgh-1958, Cambridge University Press, 154-181.

[20] Lanczos C (1966), Boundary value problems and orthogonal expansions. *SIAM J. Appl. Math*., **14**(4): 831-863

[21] Lanczos C (1997), *Linear Differential Operators*, Dover, New York.

[22] Moore EH (1920), On the reciprocal of the general algebraic matrix. *Bull. Am. Math. Soc.,* **26 (9) :** 394-395.

[23] Moler CB and Morrison D (1983), Singular value analysis of cryptograms. *Am. Math. Monthly,* **90** : 78-87.

[24] Nakos G and Joyner D (1998), *Linear algebra with applications.* Chap. 8, Brooks/Cole Pub. Co, New York.

[25] Penrose R (1955), A generalized inverse for matrices. *Proc. Camb. Phil. Soc.,* **51**: 406-413.

[26] Picard E (1910), Sur un theorem general relative aux integrals de premier espéce et sur quelques problémes de physique mathématique. *Rend. Circ. Mat. Palermo,* **25**: 79-97.

[27] Schmidt E (1907), Zur theorie der linearen und nichtlinearen integralgleichungen. *Teil 1, Mathematische Annalen Bd.* **63:** 433-476 and **64:** 161-174.

[28] Smithies F (1963), Linear differential operators. *The Mathematical Gazette* **47**: 265-266.

[29] Stewart GW (1993), On the early history of the SVD. *SIAM Review,* **35**: 551-566

[30] Sylvester JJ (1889), Sur la réduction biorthogonale d'une forme linéo-linéaire á sa forme cannonique. *Compt. Rend. Acad. Sci. Paris,* **108:** 651-653.

[31] Sylvester JJ (1889), On the reduction of a bilinear quantic of the nth order to the form of a sum of n products. *Messenger of Mathematics,* **19**: 42-46.

[32] Weyr E (1885), Répartition des matrices en espaces et formation de toutes les espaces. *C. R. Acad. Sci. Paris,***100:** 966-969.

[33] Weyr E (1890), Zur theorie der bilinearen formen. *Mon. für Mathematik und Physik,* **1** : 163-236.

[34] Yanai H, Takeuchi K and Takane Y (2011), *Projection matrices, generalized inverse matrices, and singular value decomposition.* Chap. 3, Springer, New York.