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A Newton Type Iterative Method with Fourth-order Convergence

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Abstract: The aim of this paper is to propose a fourth-order Newton type iterative method for solving nonlinear equations in a single variable. We obtained this method by combining the iterations of contra harmonic Newton's method with secant method. The proposed method is free from second order derivative. Some numerical examples are given to illustrate the performance and to show this method's advantage over other compared methods.

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1. Introduction

Solving single variable nonlinear equations efficiently is one of the most important problems in numerical analysis and has many applications in all fields of science and engineering. In most of the cases, it is not possible to solve these equations analytically. In those situations where an analytic solution cannot be obtained or it is difficult to obtain, numerical iterative methods are employed to get approximate solution of nonlinear equations. To find a single root α of nonlinear equation $f(x) = 0$, where $f: D \subset \boldsymbol{R} \to \boldsymbol{R}$ is a scalar function on an open interval D, two best known and the most widely used iterative methods for solving nonlinear equations are Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

and the secant method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \tag{2}$$

The order of convergence of Newton method is 2 and secant method is 1.62 [2, 3].

In the last two decades, large number of researchers have been working in the area of root finding problems of single variable nonlinear equations and they have proposed numerous effective iterative methods (see [1,4,5,6,7,8,9]). Weerakoon and Fernando [9] used Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{3}$$

and approximated the integral by trapezoidal rule that is

$$\int_{x_n}^{x} f'(t)dt = \frac{(x-x_n)}{2}[f'(x_n) + f'(x)]$$

and then obtained following iterative method as a variant of Newton method:

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n)+f'(x_n^*)]} \tag{4}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

This method is known as trapezoidal Newton's method. The method (4) can be written as-

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{[f'(x_n)+f'(x_n^*)]}{2}} \tag{5}$$

The method (4) is also called arithmetic mean Newton's method because this variant of Newton method can be viewed as values obtained by using arithmetic mean of $f'(x_n)$ and $f'(x_n^*)$ instead of $f'(x_n)$ in Newton method (1).

Ababneh [1] used the contra harmonic mean instead of the arithmetic mean in (5) and he got new iterative method

$$x_{n+1} = x_n - f(x_n)\frac{[f'(x_n)+f'(x_n^*)]}{f'^2(x_n)+f'^2(x_n^*)}, \tag{6}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

This method is called contra harmonic Newton method and the order of convergence of this method is 3.

2. The Iterative Method and Convergence Analysis

As the aim of this paper, we purpose the following method in which the iteration are preformed alternatively from method (6) and the secant method:

$$x_{n+1=} \tilde{x}_n - \frac{\tilde{x}_n - x_n}{f(\tilde{x}_n) - f(x_n)} f(\tilde{x}_n), \tag{7}$$

where $\quad \tilde{x}_n = x_n - f(x_n) \frac{[f'(x_n) + f'(x_n^*)]}{f'^2(x_n) + f'^2(x_n^*)}, \qquad (8)$

with $\quad x_n^* = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (9)$

Precisely, we have proved the following theorem for convergence analysis.

Theorem: Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (7)-(10) is convergent with order of convergence 4.

Proof. Let e_n and $\tilde{e}_n$ be the error in x_n and $\tilde{x}_n$ respectively. Then $x_n = \alpha + e_n$ and $\tilde{x}_n = \alpha + \tilde{e}_n$. Ababneh [1] proved that the error equation of (9) as

$$\tilde{e}_n = \left(2c_2^2 + \frac{1}{2} c_3\right) e_n^3 + O(e_n^4) \quad = \mathrm{K} e_n^3 + O(e_n^4) \tag{10}$$

where $\quad c_{j=} \frac{1}{j\,!} \frac{f_{(\alpha)}^{j}}{f_{(\alpha)}'}, \quad j = 2,3, \ldots.$ and $K = 2c_2^2 + \frac{1}{2} c_3$

Using Taylor's theorem, we get

$$f(x_n) = f(\alpha + e_n)$$

$$= f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \frac{e_n^3}{3!} f'''(\alpha) + \frac{e_n^4}{4!} f'^{v}(\alpha) + \cdots$$

$$= e_n f'(\alpha)[1 + c_2 e_n + c_3 e_n^2 + O(e_n^3)] \tag{11}$$

and $\quad f(\tilde{x}_n) = f(\alpha + \tilde{e}_n)$

$$= f(\alpha) + \tilde{e}_n f'(\alpha) + \frac{\tilde{e}_n{}^2}{2!} f''(\alpha) + \cdots$$

$$= \tilde{e}_n f'(\alpha) + \frac{\tilde{e}_n{}^2}{2!} f''(\alpha) + \cdots$$

$$= K e_n^3 f'(\alpha) + O(e_n^4). \tag{12}$$

Since $f(\alpha) = 0$, as α is the root of $f(x)$.

Thus,

$$f(\tilde{x}_n) - f(x_n) = e_n f'(\alpha)[-1 - c_2 e_n + (K - c_3)e_n^2 + O(e_n^3)]$$

$$= -e_n f'(\alpha)[1 + c_2 e_n + (c_3 - K)e_n^2 + O(e_n^3)]$$

and $$\frac{1}{f(\tilde{x}_n) - f(x_n)} = \frac{1}{-e_n f'(\alpha)}[1 + c_2 e_n + (c_3 - K)e_n^2 + O(e_n^3)]^{-1}$$

$$= \frac{1}{-e_n f'(\alpha)}[1 - \{c_2 e_n + (c_3 - K)e_n^2 + O(e_n^3)\}$$

$$+\{1 + c_2 e_n + (c_3 - K)e_n^2 + O(e_n^3)\}^2 + \ldots]$$

$$= \frac{1}{-e_n f'(\alpha)}[1 - c_2 e_n + (K - c_3 + c_2^2)e_n^2 + O(e_n^3)] \quad (13)$$

Also, we have $$\tilde{x}_n - x_n = \alpha + \tilde{e}_n - (\alpha + e_n)$$

$$= \tilde{e}_n - e_n$$

$$= e_n^3 - e_n + O(e_n^4) \quad (14)$$

From (12), (13) and (14), we get

$$\frac{\tilde{x}_n - x_n}{f(\tilde{x}_n) - f(x_n)} f(\tilde{x}_n) = [K\, e_n^3 + O(e_n^4)]\, [1 - c_2 e_n + (K - c_3 + c_2^2)e_n^2 + O(e_n^3)]$$

$$= K e_n^3 + O(e_n^4)$$

Thus, error equation of (7) is given by

$$e_{n+1} = \tilde{e}_n - [K e_n^3 + O(e_n^4)]$$

$$= \mathrm{K}\, e_n^3 + O(e_n^4) - K e_n^3 + O(e_n^4)]$$

$$= O(e_n^4),$$

which prove that order of convergence of method (7)-(9) is of order 4.

3. Numerical Examples

In order to check the performance of the introduced fourth-order method, we have presented numerical results on some test functions. We have also compared the results of this method with Newton method (NM), Weerakoon and Fernando (W-F) method and contra harmonic Newton method. Numerical computations have been performed on Matlab software. We have used the stopping criteria $|x_{n+1} - x_n| < \varepsilon$, where $\varepsilon = (10)^{-14}$ and $|f(x_n)| < \delta$ where $\delta = (10)^{-15}$ for the iterative process of our results.

The test functions and their roots α which are used as numerical examples are given below:

(i) $f_1 = x^3 + 4x^2 - 10$, $\alpha = 1.365230013414097$

(ii) $f_2 = (x - 1)^8 - 1$, $\alpha = 2$

(iii) $f_3 = cosx - xe^x + x^2$, $\alpha = 0.639154069332008$

(iv) $f_4 = sin^2x - x^2 + 1$, $\alpha = 1.404491648215341$

(v) $f_5 = x^2 - e^x - 3x + 2$, $\alpha = 0.2575302854398608$

Table 1: Comparison table for the function $f_1 = x^3 + 4x^2 - 10$, taking initial guess $x_0 = 1$.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton	1	1.454545454545455	0.454545454545455	1.540195341848236
	2	1.368900401069519	0.085645053475936	0.060719688639942
	3	1.365236600202116	0.003663800867403	0.000108770610424
	4	1.365230013435367	0.000006586766749	0.000000000351239
	5	1.365230013414097	0.000000000021270	0.000000000000000
Trapezoidal Newton's	1	1.345024237239806	0.345024237239806	0.330369040261342
	2	1.365227728691384	0.020203491451578	0.000037728495677
	3	1.365230013414097	0.000002284722713	0.000000000000000
Contra harmonic mean Newton's	1	1.326092820189911	0.326092820189911	0.633947708152762
	2	1.365197720317173	0.039104900127263	0.000533260354388
	3	1.365230013414080	0.000032293096906	0.000000000000286
	4	1.365230013414097	0.000000000000017	0.000000000000000
Present (7)-(9)	1	1.373441267296347	0.373441267296347	0.136142115306203
	2	1.365230014536936	0.008211252759411	0.000000018541892
	3	1.365230013414097	0.000000001122839	0.000000000000000

Table 2: Comparison table for the function $f_2 = (x - 1)^8 - 1$, taking initial guess $x_0 = 3$.

Method	n	x_n	$\lvert x_n - x_{n-1} \rvert$	$\lvert f(x_n) \rvert$
Newton	1	2.750976562500000	0.249023437500000	87.357346190860710
	2	2.534581615819526	0.216394946680474	29.755297235115897
	3	2.348995976046720	0.185585639772806	9.966934095828560
	4	2.195747198046065	0.153248778000655	3.179409976051812
	5	2.082041836760382	0.113705361285683	0.879110950859817
	6	2.018764916659598	0.063276920100784	0.160357585160917
	7	2.001166173395949	0.017598743263649	0.009367555001261
	8	2.000004743257317	0.001161430138632	0.000037946688497
	9	2.000000000078744	0.000004743178573	0.000000000629949
	10	2.000000000000000	0.000000000078744	0.000000000000000
Trapezoidal Newton's	1	2.642780601144118	0.357219398855882	52.044031781287146
	2	2.354990027463031	0.287790573681086	10.362889700504827
	3	2.138737278626852	0.216252748836179	1.827406853620104
	4	2.020373440786471	0.118363837840382	0.175095510323507
	5	2.000119480784913	0.020253960001557	0.000956246093262
	6	2.000000000026847	0.000119480758066	0.000000000214779
	7	2.000000000000000	0.000000000026847	0.000000000000000
Contra harmonic mean Newton's	1	2.699505988924965	0.300494011075035	68.595569774589435
	2	2.446809725767508	0.252696263157457	18.199563543679137
	3	2.239873461530432	0.206936264237076	4.584945186476709
	4	2.086661678930914	0.153211782599518	0.944262123369496
	5	2.010026436999853	0.076635241931061	0.083083478739881
	6	2.000026415906221	0.010000021093632	0.000211346789200
	7	2.000000000000516	0.000026415905705	0.000000000004128
	8	2.000000000000000	0.000000000000516	0.000000000000000
Present (7)-(9)	1	2.588926224921406	0.411073775078594	39.628413545152938
	2	2.272225422328508	0.316700802592898	5.862977216933422
	3	2.062754438656234	0.209470983672275	0.627284246036169
	4	2.000891038518675	0.061863400137558	0.007150578400151
	5	2.000000000061337	0.000891038457338	0.000000000490697
	6	2.000000000000000	0.000000000061337	0.000000000000000

Table 4: Comparison table for the function $f_3 = cosx - xe^x + x^2$, taking initial guess $x_0 = 1$.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton	1	0.724644697567095	0.275355302432905	0.221820009620489
	2	0.644658904870270	0.079985792696824	0.013402585003872
	3	0.639177807467281	0.005481097402990	0.000057481709000
	4	0.639154096773051	0.000023710694230	0.000000001069179
	5	0.639154096332008	0.000000000441043	0.000000000000000
Trapezoidal Newton's	1	0.665881945014898	0.334118054985102	0.066172545899518
	2	0.639169572742496	0.026712372272402	0.000037518438781
	3	0.639154096332011	0.000015476410485	0.000000000000008
	4	0.639154096332008	0.000000000000003	0.000000000000000
Contra harmonic mean Newton's	1	0.680435718288194	0.319564281711806	0.103390816892531
	2	0.639250900896522	0.041184817391673	0.000234691879059
	3	0.639154096333320	0.000096804563201	0.000000000003182
	4	0.639154096332008	0.000000000001313	0.000000000000000
Present (7)-(9)	1	0.649689059626464	0.350310940373536	0.025751201751007
	2	0.639154110108977	0.010534949517487	0.000000033398190
	3	0.639154096332008	0.000000013776969	0.000000000000000

Table 5 : Comparison table for the function $f_4 = sin^2x - x^2 + 1$, taking initial guess $x_0 = 1$.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton	1	1.649190196932272	0.649190196932272	0.725961325382220
	2	1.439042347687187	0.210147849245085	0.088101775402360
	3	1.405385086160459	0.033657261526728	0.002219488277199
	4	1.404492272936243	0.000892813224217	0.000001550853411
	5	1.404491648215647	0.000000624720595	0.000000000000759
	6	1.404491648215341	0.000000000000306	0.000000000000000
Trapezoidal Newton's	1	1.311567755348155	0.311567755348155	0.214082403902797
	2	1.403843587357776	0.092275832009621	0.001607976570500
	3	1.404491648036039	0.000648060678263	0.000000000445113
	4	1.404491648215341	0.000000000179302	0.000000000000000

Contra harmonic mean Newton's	1 2 3 4	1.245238231250271 1.397048894592266 1.404491114013924 1.404491648215341	0.245238231250271 0.151810663341996 0.007442219421658 0.000000534201417	0.347085646983947 0.018368771038801 0.000001326139939 0.000000000000000
Present (7)-(9)	1 2 3	1.481032008570244 1.404518413202386 1.404491648215341	0.481032008570244 0.076513595367858 0.000026764987044	0.201491824657889 0.000066444746114 0.000000000000000

Table 6: Comparison table for the function $f_5 = x^2 - e^x - 3x + 2$, taking initial guess $x_0 = 0$.

Method	n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
Newton	1 2 3 4	0.250000000000000 0.257524945045740 0.257530285437195 0.257530285439861	0.250000000000000 0.007524945045740 0.000005340391455 0.000000000002665	0.028474583312259 0.000020179599371 0.000000000010071 0.000000000000000
Trapezoidal Newton's	1 2 3	0.256936468335819 0.257530285432059 0.257530285439861	0.256936468335819 0.000593817096240 0.000000000007802	0.002243963693351 0.000000000029480 0.000000000000000
Contra harmonic mean Newton's	1 2 3	0.256738822192933 0.257530285417054 0.257530285439861	0.256738822192933 0.000791463224121 0.000000000022807	0.002990900075557 0.000000000086181 0.000000000000000
Present (7)-(9)	1 2	0.257509005898130 0.257530285439861	0.257509005898130 0.000021279541731	0.000080408534813 0.000000000000000

4. Conclusion

From section 2, it is evident that when the iterations are performed alternately from third order contra harmonic Newton's method and secant method, we get a fourth –order iterative method. This result is also supported by numerical examples. From tables, we observed that the iterative method (7)-(9) takes less number of iterations as compared to Newton method, trapezoidal Newton's method as well as contra harmonic Newton's method. Moreover, the method which we have derived is free from higher order derivatives and in most of the cases, this method requires less or equal number of functions evaluation as required in other compared methods. Thus, the method (7)-(9) is not only faster but also requires no more effort than the methods mention here.

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