A Short Note on Hyper-geometric Expression

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Abstract: In this short note, we exhibit an elementary deduction of the Borwein-Choi-Pigulla relation for \( \log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} \) in terms of the Gauss hyper-geometric function.

Keywords: Hyper-geometric function, Borwein-Choi-Pigulla’s formula.

1. Introduction

In this paper, we briefly discuss on the hyper-geometric expression of \( \log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} \). The researchers Borwein-Choi-Pigulla [3] employed continued fractions to obtain the following identity [2]:

\[
\log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} = -\frac{(-z)^n}{n} \binom{n}{1} \binom{n}{1} \binom{n}{1} = -\frac{(-z)^n}{n} \binom{n}{1} \binom{n}{1} \binom{n}{1} = -\frac{(-z)^n}{n} \binom{n}{1} \binom{n}{1} \binom{n}{1}.
\]

(1)

Here we show a simple deduction of (1) via the techniques of [4, 5] to convert a summation into a hyper-geometric function.

2. Borwein-Choi-Pigulla Formula

We have the known Taylor series for the logarithm function:

\[
\log(1+z) - z = -\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = -\frac{z^2}{2} \left( 1 - \frac{2z^2}{3} + \frac{2z^3}{4} - \frac{2z^4}{5} + \cdots \right),
\]

(2)

and it is simple to apply the techniques of [4, 5] to convert a summation into a Gauss hyper-geometric function, in fact,

\[
u_0 = 1 \quad \text{and} \quad \frac{u_{k+1}}{u_k} = \frac{(k+2) (k+1) (-z)}{(k+3) (k+1)},
\]

therefore, we have

\[
\log(1+z) - z = -\frac{z^2}{2} \binom{2}{1} \binom{1}{1} \binom{1}{1} = -\frac{z^2}{2} \binom{2}{1} \binom{1}{1} \binom{1}{1}.
\]

Similarly, from (2):
\[
\log (1 + z) - z + \frac{z^2}{2} = \frac{z^3}{3} \sum_{k=0}^{\infty} \frac{3(-1)^k z^k}{k+3} \\
= \frac{z^3}{3} \, _2F_1(3, 1; 4; -z)
\]

and in general:
\[
\log (1 + z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} = -\frac{(-z)^n}{n} \sum_{k=0}^{\infty} \frac{n (-z)^k}{k+n} \\
= -\frac{(-z)^n}{n} \, _2F_1(n, 1; n + 1; -z)
\]
in harmony with the result (1) due to Borwein-Choi-Pigulla.

As always, a formula for \( \log \) leads correspondingly to one for \( \arctan \) [2]:
\[
\arctan z - \sum_{k=0}^{n-1} \frac{(-1)^k z^{2k+1}}{2k+1} = \frac{(-1)^n z^{2n+1}}{2n+1} \, _2F_1 \left( n + \frac{1}{2}, 1; n + \frac{3}{2}; -z^2 \right). \tag{3}
\]

3. Conclusion

Our procedure shows that the techniques of [4, 5] are very useful to translate a summation in terms of hyper-geometric functions, and thus to give simple proofs for several important formulae in the literature [1].

References


