



# Gibbs-Wilbraham Phenomenon in Square Wave Function

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**Abstract:** The aim of this paper is to show the fluctuation on overshoots of particular term due to change in magnitude of discontinuity in the square wave function represented in Fourier series and also is to address the mathematical relation between those parameters. Along with that, some graphical plots of different terms are given to illustrate the results and to interpolate data with smooth curves using Cubic spline interpolation in MATLAB.

**Keywords:** Fourier series, Gibbs-Wilbraham phenomenon, overshoots, square wave function, interpolation

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## 1. Introduction

Several complicated functions may be represented in power series and that is not the only way of expressing those functions, and that can be done by the combinations of functions as a sum of sine and cosine terms. Such a representation of functions is called the Fourier series. It has several advantages and may be used to represent some functions for which a Taylor series expansion is not possible. A function  $f(x)$  must satisfy the conditions called Dirichlet conditions in order to get its Fourier form. Fourier series has a wide application on different physical situations including the transmission of an input signal by an electronic circuit, the scattering of light by a diffraction grating and so on [5]. If the function is odd, it can be represented by sine series and similar with the case for an even it is done by proper sum of the cosine series. So, all functions may be written as the sum of an odd and an even part as follows.

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

All the terms of a Fourier series are mutually orthogonal. The Fourier series expansion of the functions  $f(t)$  is conventionally written as below

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[ a_r \cos\left(\frac{2\pi r t}{L}\right) + b_r \sin\left(\frac{2\pi r t}{L}\right) \right]$$

where,  $a_0$ ,  $a_r$ ,  $b_r$  are constants called the Fourier coefficients. We can evaluate the coefficients for any particular functions  $f(t)$  of period  $L$  and are given by

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(t) \cos\left(\frac{2\pi r t}{L}\right) dt$$

$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(t) \sin\left(\frac{2\pi r t}{L}\right) dt$$

where  $x_0$  is arbitrary but is often taken as  $0$  or  $-L/2$ .

Since our work is concerned with the square wave functions, the Fourier series expansion of the function can be obtained as below.

The particular unit square wave function physically might represent the input to an electrical circuit that switches between a high and a low state with the time period  $T$ . Mathematically, the unit square wave can be represented by

$$f(t) = \begin{cases} -1 & \text{for } \frac{-1}{2}T \leq t < 0 \\ +1 & \text{for } 0 \leq t < \frac{1}{2}T \end{cases}$$

Firstly the given function is an odd so the series will contain the sine terms (from symmetry consideration). Hence to evaluate the coefficient in the series, we have

$$b_r = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi r t}{T}\right) dt$$

$$= \frac{4}{T} \int_0^{\frac{T}{2}} \sin\left(\frac{2\pi r t}{T}\right) dt$$

$$= \frac{2}{\pi r} [1 - (-1)^r]$$

From the above equation the sine coefficients are zero if  $r$  is even and equal to  $4/(\pi r)$  if  $r$  is odd. So the Fourier series for the unit square wave function may be written as below

$$f(t) = \frac{4}{\pi} \left( \sin(\omega t) + \frac{\sin(3\omega t)}{3} + \frac{\sin(5\omega t)}{5} + \frac{\sin(7\omega t)}{7} + \dots \right)$$

where  $\omega = 2\pi/T$  is called the angular frequency.

Since the above square wave function is discontinuous in the required range and however, the series itself, does not produce a discontinuous function. So at a point of finite discontinuity  $x_d$ , the Fourier series converges to a point below

$$\frac{1}{2} \lim_{\epsilon \rightarrow 0} [f(x_d + \epsilon) + f(x_d - \epsilon)] = \left[ \frac{f(x_d^+) + f(x_d^-)}{2} \right]$$

During the transition of series from the  $f(x_d^-)$  to  $f(x_d^+)$ , the function will overshoot its value.

As more terms are included the overshoot moves in position arbitrarily close to the discontinuity [4]. The ‘‘overshoot’’ on either side of a jump does not disappear as we add more and more terms of the series rather it becomes narrower spike of height equal to about 9% of the jump [1]. The  $n$ th partial sum of the Fourier series has large oscillations near the jump and has the peak value of

overshoot at the first overshoot and gradually it decreases but never die out but approaches a finite limit. This phenomenon was discovered by H Wilbraham (1848) and rediscovered by JW Gibbs (1899) [2]. So Gibb's phenomenon deals with "overshoot" in the convergence of the partial sums of certain Fourier series in neighborhood of a discontinuity of the function being expanded [3]. This sort of behavior was also observed by experimental physicists, but was believed to be due to imperfections in the measuring apparatuses. The size of the Gibbs overshoot is proportional to the magnitude of the discontinuity

$$\delta(n, m, p) \propto D$$

where,  $\delta(n, m, p)$  is the size of the overshoots for  $n^{\text{th}}$  terms with amplitude 'm' for different overshoots 'p' and 'D' is the magnitude of discontinuity. And our work is concerned with the mathematical relation between the Gibbs overshoots and magnitude of discontinuity for several terms.

## 2. Relation Between $\delta, n, m$ and $p$

Before approaching to general form, let's address the relations for the particular or simpler cases. As given in unit 1, the Fourier series expansion of square wave function of unit amplitude ( $m = 1$ ) is

$$f(t) = \frac{4}{\pi} \left( \sin(\omega t) + \frac{\sin(3\omega t)}{3} + \frac{\sin(5\omega t)}{5} + \frac{\sin(7\omega t)}{7} + \dots \right)$$

Differentiating the above equation for the calculation of stationary points within the interval 0 and T/2. Due to its periodicity we can manipulate or predict its behavior by analyzing the function within the given interval. So, the differentiation of above function is given below,

$$f'(t) = \frac{4}{\pi} (\omega \cos(\omega t) + \omega \cos(3\omega t) + \omega \cos(5\omega t) + \omega \cos(7\omega t) + \dots)$$

Equating the above equation to 0, we get,

$$\frac{4}{\pi} (\omega \cos(\omega t) + \omega \cos(3\omega t) + \omega \cos(5\omega t) + \omega \cos(7\omega t) + \dots) = 0$$

$$\text{or, } \cos(\omega t) + \cos(3\omega t) + \cos(5\omega t) + \cos(7\omega t) + \dots = 0$$

$$\text{or, } \sum_{r=0}^{n-1} \cos(2r+1)\omega t = 0$$

Taking for the first term ( $n=1$ ), we get the following result from the above equation,

I) For  $n = 1$ ,

$$\cos \omega t = 0 \Rightarrow \omega t = \pi / 2$$

$$\therefore t = \left( \frac{T}{4} \right)$$

The functional value at that point is,

$$f\left(\frac{T}{4}\right) = \frac{4}{\pi} \sin\left(\omega \cdot \frac{T}{4}\right) = \frac{4}{\pi} \sin\left(\frac{\pi}{2}\right) \approx 1.27$$

So, for the size of the overshoot for the  $n=1$ ,  $m=1$  and  $p=1$ , we have

$$\delta(n, m, p) = \delta(1, 1, 1) = \left[ f\left(\frac{T}{4}\right) - 1 \right] \approx 0.27$$

II) For  $n = 2$ ,

$$\cos(\omega t) + \cos 3(\omega t) = 0$$

$$\cos(\omega t) = -\left[ 4 \cos^3(\omega t) - 3 \cos(\omega t) \right]$$

$$\cos(\omega t) (4 \cos^2(\omega t) - 2) = 0$$

$$\text{either, } \cos(\omega t) = 0 \Rightarrow t_2 = \left(\frac{T}{4}\right)$$

$$\text{or, } \cos(\omega t) = \cos\left(\frac{\pi}{4}\right) \Rightarrow t_1 = \left(\frac{T}{8}\right)$$

So, for the first overshoot's size we take the  $t_1$  and take its functional value as before.

The size of the overshoot for the  $n = 2$ ,  $m = 1$  and  $p = 1$ , we have

$$\delta(n, m, p) = \delta(2, 1, 1) = \left[ f\left(\frac{T}{8}\right) - 1 \right] < \delta(1, 1, 1)$$

Here,  $t_2$  gives the overshoot below the amplitude of the square wave function.

Hence, if the process is repeated for the several terms we obtained the stationary points from where we can have the size of overshoots. For  $n^{\text{th}}$  term, there are  $(2n-1)$  stationary points within the addressed interval and also ' $n$ ' numbers of overshoots over the amplitude ( $m=1$ ) and ' $n-1$ ' below it. Our results is concerned with the overshoots above the amplitude within the interval 0 and  $T/4$ . Here, oscillation shows the periodic behavior after the interval  $T/4$  up to the  $T/2$ .

1) For odd terms, taking  $n = 3$ ,

The stationary points for the overshoots above the amplitude will be at,

So, for  $p = 1$  (first overshoot),  $t_1 = \frac{T}{12}$  and the respective size of the overshoot is

$$\delta(n, m, p) = \delta(3, 1, 1) = \left[ \frac{4}{\pi} \left\{ \sin\left(\frac{\pi}{6}\right) + \frac{\sin 3\left(\frac{\pi}{6}\right)}{3} + \frac{\sin 5\left(\frac{\pi}{6}\right)}{5} \right\} - 1 \right]$$

And for  $p=2$  (second overshoot),  $t_2 = \frac{3T}{12}$  and the respective size of the overshoot is

$$\delta(n, m, p) = \delta(3, 1, 2) = \left[ \frac{4}{\pi} \left\{ \sin\left(\frac{3\pi}{6}\right) + \frac{\sin 3\left(\frac{3\pi}{6}\right)}{3} + \frac{\sin 5\left(\frac{3\pi}{6}\right)}{5} \right\} - 1 \right]$$

From the above value of overshoots,  $\delta(3, 1, 1) > \delta(3, 1, 2)$

Similarly, if we continue with the odd terms, there we can obtain the pattern in which the size of overshoots are calculated. If the term is taken  $n = 5$ , there exist three overshoots over the amplitude within the interval  $0$  and  $T/4$  at  $T/20$ ,  $3T/20$  and  $5T/20$  for  $p = 1$ ,  $p = 2$  and  $p = 3$  respectively.

Hence, In general for odd term, there are  $(n+1)/2$  no. of overshoots over the amplitude within the interval  $0$  and  $T/4$  at  $T/4n$ ,  $3T/4n$ ,  $5T/4n$ ,  $7T/4n$  ... for  $p = 1$ ,  $p = 2$ ,  $p = 3$ ,  $p = 4$  ... respectively.

So, the general formula for the size of overshoots is given below,

$$\delta(n, 1, p) = \left[ \frac{4}{\pi} \left\{ \frac{\sin\left(\frac{(2p-1)\pi}{2n}\right)}{1} + \frac{\sin 3\left(\frac{(2p-1)\pi}{2n}\right)}{3} + \frac{\sin 5\left(\frac{(2p-1)\pi}{2n}\right)}{5} + \dots \right\} - 1 \right]$$

$$\delta(n, 1, p) = \left[ \frac{4}{\pi} \sum_{r=0}^{n-1} \left\{ \frac{\sin\left(\frac{(2r+1)(2p-1)\pi}{2n}\right)}{(2r+1)} \right\} - 1 \right]$$

Since, it holds for the square wave function with unit amplitude ( $m = 1$ ), more general case will be, if the function is taken with the amplitude ‘ $m$ ’ having the magnitude of discontinuity ‘ $D$ ’ and at that situation the function is represented as below,

$$f(t) = \begin{cases} u & \text{for } -\frac{1}{2}T \leq t < 0 \\ v & \text{for } 0 \leq t < \frac{1}{2}T \end{cases}$$

where a) If ‘ $u$ ’ and ‘ $v$ ’ are of opposite sign,  $D = (|u| + |v|)$  is the magnitude of discontinuity,

$m = \frac{(|u| + |v|)}{2}$  is the amplitude for the square wave function.

b) If ‘ $u$ ’ and ‘ $v$ ’ are of same sign,  $D = (|u| - |v|)$  is the magnitude of discontinuity,

$m = \frac{(|u| - |v|)}{2}$  is the amplitude for the square wave function for  $p = 1, 2, 3 \dots \left(\frac{n+1}{2}\right)$ . And the

size of Gibbs overshoots are given as below:

$$\delta(n, m, p) = \left[ \frac{4m}{\pi} \sum_{r=0}^{n-1} \left\{ \frac{\sin \left( \frac{(2r+1)(2p-1)\pi}{2n} \right)}{(2r+1)} \right\} - m \right]$$

2) For even terms, taking  $n = 4$ ,

The stationary points for the overshoots above the amplitude will be at,

So, for  $p = 1$  (first overshoot),  $t_1 = \frac{T}{16}$  and the respective size of the overshoot is

$$\delta(n, m, p) = \delta(4, 1, 1) = \left[ \frac{4}{\pi} \left\{ \sin \left( \frac{\pi}{8} \right) + \frac{\sin 3 \left( \frac{\pi}{8} \right)}{3} + \frac{\sin 5 \left( \frac{\pi}{8} \right)}{5} + \frac{\sin 7 \left( \frac{\pi}{8} \right)}{7} \right\} - 1 \right]$$

And for  $p = 2$  (second overshoot),  $t_2 = \frac{3T}{16}$  and the respective size of the overshoot is

$$\delta(n, m, p) = \delta(4, 1, 2) = \left[ \frac{4}{\pi} \left\{ \sin \left( \frac{3\pi}{8} \right) + \frac{\sin 3 \left( \frac{3\pi}{8} \right)}{3} + \frac{\sin 5 \left( \frac{3\pi}{8} \right)}{5} + \frac{\sin 7 \left( \frac{3\pi}{8} \right)}{7} \right\} - 1 \right]$$

From the above value of overshoots,  $\delta(4, 1, 1) > \delta(4, 1, 2)$ .

Similarly, if we continue with the even terms as we have done before with the odd one, there we can obtain the pattern in which the size of overshoots are calculated. If the term is taken  $n = 6$ , there exist three overshoots over the amplitude within the interval  $0$  and  $T/4$  at  $T/32$ ,  $3T/32$  and  $5T/32$  for  $p = 1$ ,  $p = 2$  and  $p = 3$  respectively.

Hence, In general for even term, there are  $n/2$  no. of overshoots over the amplitude within the

interval  $0$  and  $T/4$  at  $\frac{T}{2^{\left(\frac{n}{2}+2\right)}}$ ,  $\frac{3T}{2^{\left(\frac{n}{2}+2\right)}}$ ,  $\frac{5T}{2^{\left(\frac{n}{2}+2\right)}}$ ,  $\frac{7T}{2^{\left(\frac{n}{2}+2\right)}}$  ... for  $p = 1$ ,  $p = 2$ ,  $p = 3$ ,  $p = 4$ , ...

respectively. So, the general formula for the size of overshoots is given below:

$$\delta(n,1,p) = \left[ \frac{4}{\pi} \left\{ \frac{\sin\left(\frac{(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{1} + \frac{\sin 3\left(\frac{(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{3} + \frac{\sin 5\left(\frac{(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{5} + \frac{\sin 7\left(\frac{(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{7} + \dots \right\} - 1 \right]$$

$$\delta(n,1,p) = \left[ \frac{4}{\pi} \sum_{r=0}^{n-1} \left\{ \frac{\sin\left(\frac{(2r+1)(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{(2r+1)} \right\} - 1 \right]$$

Since, it holds for the square wave function with unit amplitude (m=1), more general case will be, if the function is taken with the amplitude ‘m’ having the magnitude of discontinuity ‘D’ and at that situation the function is represented as below:

$$f(t) = \begin{cases} u \text{ for } -\frac{1}{2}T \leq t < 0 \\ v \text{ for } 0 \leq t < \frac{1}{2}T \end{cases}$$

where

1) If ‘u’ and ‘v’ are of opposite sign,  $D = (|u| + |v|)$  is the magnitude of discontinuity,

$m = \frac{(|u| + |v|)}{2}$  is the amplitude for the square wave function.

2) If ‘u’ and ‘v’ are of same sign,  $D = (|u| - |v|)$  is the magnitude of discontinuity,

$m = \frac{(|u| - |v|)}{2}$  is the amplitude for the square wave function for  $p = 1, 2, 3 \dots \left(\frac{n}{2}\right)$ . And the size

of Gibbs overshoots are given as below:

$$\delta(n,m,p) = \left[ \frac{4m}{\pi} \sum_{r=0}^{n-1} \left\{ \frac{\sin\left(\frac{(2r+1)(2p-1)\pi}{2^{\left(\frac{n}{2}+1\right)}}\right)}{(2r+1)} \right\} - m \right]$$

3. Graphs and Data Plot

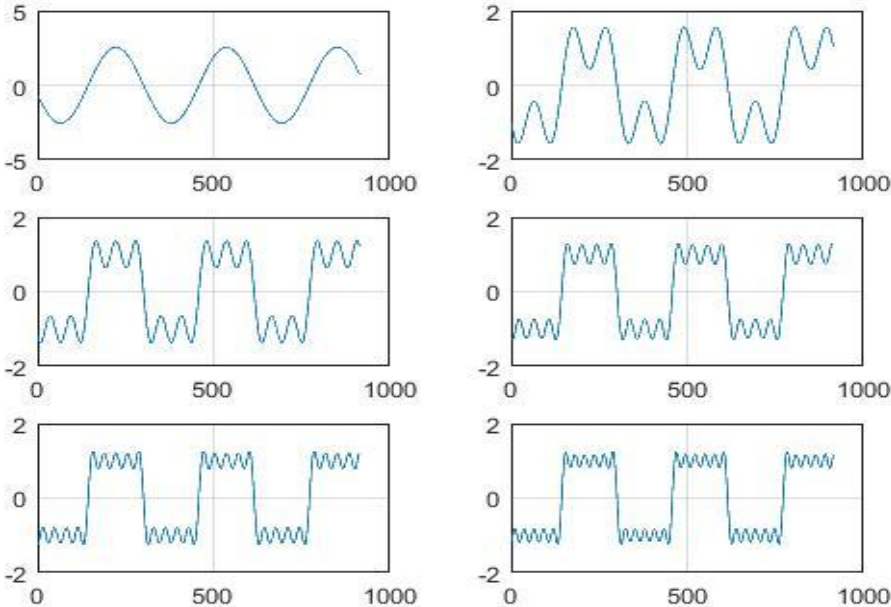


Fig 1: Fourier expansion of square wave function for n = 1, n = 2, n = 3, n = 4, n = 5 and n = 6

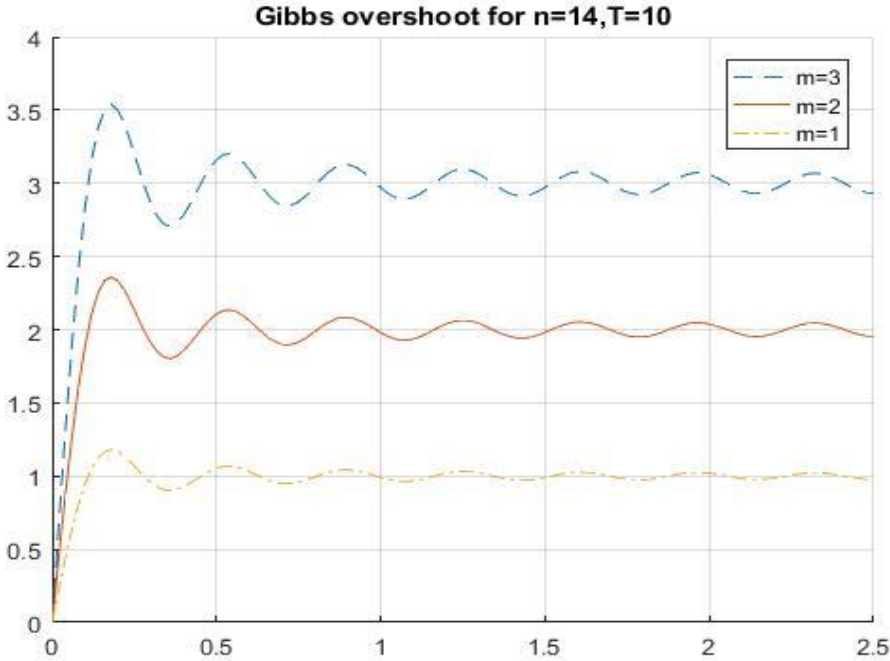


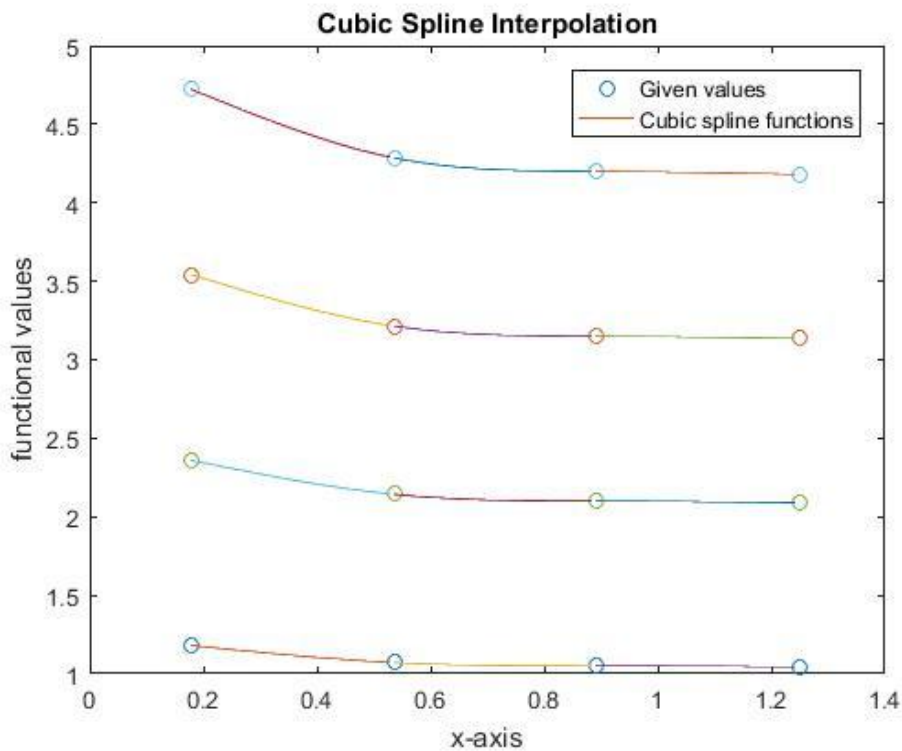
Fig 2: Variation in Size of Gibbs overshoots due to amplitude of square wave function



**Data for the cubic spline function:**

Table 1: Taking x co-ordinate for the peak points of consecutive overshoots above the amplitude (n=7, T=5) and its functional values.

	x	0.17857	0.5357	0.892857	1.25
<b>m=1</b>	<b>F<sub>1</sub>(x)</b>	1.18	1.07155	1.0501	1.045
<b>m=2</b>	<b>F<sub>2</sub>(x)</b>	2.36	2.1431	2.100	2.09
<b>m=3</b>	<b>F<sub>3</sub>(x)</b>	3.542	3.21465	3.15	3.13574
<b>m=4</b>	<b>F<sub>4</sub>(x)</b>	4.7227	4.2862	4.2	4.181



**Fig 3: Decrement rate on size of overshoots (n = 7) for different amplitude of square wave function m = 4, m = 3, m = 2 and m = 1 (up to down)**

**4. Results and Discussion**

From above obtained results, we got two cases (even and odd) relating Gibbs overshoots and magnitude of discontinuity for different terms and for different overshoots above the amplitude. And using MATLAB programming platform, various graphical plots are obtained as shown in fig above. Through the use of cubic spline interpolation methods we approximate the peak points of the Gibbs overshoots for particular terms.

## 5. Conclusion

There exists relation between overshoots sizes with the magnitude of discontinuity or proportional with it. And also there we attained the particular decrement rate in sizes for consecutive overshoots for definite terms. There are many corners to be studied in this area which will be our future works.

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