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Direct Method for the Determination of Coefficients of Characteristic Equation of a MDOF System

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Abstract: Dynamic response of any single degree of freedom (SDOF) vibratory system is studied by evaluating its natural frequencies whereas that of any multi degree of freedom (MDOF) vibratory system is studied by evaluating its natural frequencies and corresponding mode shapes. Efficient method to determine the natural frequencies and mode shape of a MDOF system is to determine its dynamic matrix and to calculate its eigen-values and eigen-vectors. As the number of degree of freedom (DOF) of the system increases, the size of the dynamic matrix increases and the use of a computer program or package become essential. Hence this paper proposes a new method to directly calculate the coefficients of characteristic equation of any degree of freedom system from which eigen-values and then natural frequencies can be determined.

Keywords: MDOF, Dynamic Matrix, Characteristic Equation, Eigen-value, Natural Frequencies.

1. Introduction

The procedure to study dynamic behavior of any physical system is to develop the mathematical model in the form of its governing equation and to solve the equation for its natural frequencies and mode shapes. Depending upon the situation, the problem of interest can be modeled as a discrete or a continuous system. Discrete system can further be modeled as a single degree of freedom or a multi-degree of freedom system. If the system of interest can be modeled as a single degree of freedom system then its response can be explained in terms of natural frequency whereas if the system is modeled as a multi degree of freedom system then its response is explained in terms of its natural frequencies and corresponding mode shapes [1, 2, 3].

Governing equation of a SDOF system appear in the form of an ordinary differential equation whereas governing equation of a MDOF system appear in the form of a set of coupled ordinary differential equations. The number of coupled differential equations will be equal to the degree of freedom of the system. Hence, the analysis of dynamical systems of several degrees of freedom is complicated by a large number of equations and many detailed computations. Therefore the problem of larger degree of freedom can be concisely written in matrix form in terms of mass matrix and stiffness matrix. By knowing the mass matrix and stiffness matrix, dynamic matrix can be determined from which eigen values and eigen vectors can be determined leading to natural frequencies and mode shapes [1, 2, 3]. Eigen-values of a dynamic matrix can be determined manually when the degree of freedom is less but it will be cumbersome when the degree of freedom increases. In this paper coefficients of characteristic equation are determined by matrix method and by comparing the coefficients of each order term, generalized polynomials are developed such that coefficients of characteristic equation can be determined directly.

2. Equation of Motion of MDOF System in Matrix Form and Characteristic Equation

2.1 Equation of Motion in Matrix Form

Consider an undamped system shown in Fig. 1 having n degrees of freedom. Let $x_1, x_2, x_3, \ldots, x_n$ are the displacements for the equilibrium positions of the respective masses at any instant.

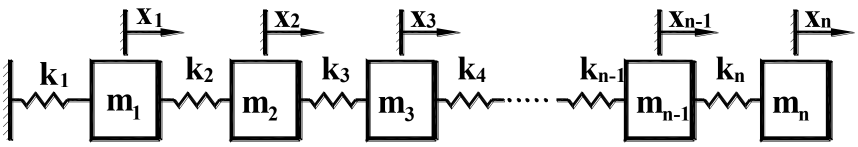

Fig. 1: A MDOF system

The differential equations for each mass separately can be written by applying Newton's second law as

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2)\\
m_2\ddot{x}_2 &= k_2(x_1 - x_2) - k_3(x_2 - x_3)\\
m_3\ddot{x}_3 &= k_3(x_2 - x_3) - k_4(x_3 - x_4)\\
\ldots\ldots &= \ldots\ldots\ldots \qquad \ldots\ldots\ldots\\
m_n\ddot{x}_n &= k_n(x_{n-1} - x_n)
\end{aligned}
$$

These equations can be arranged in the following forms,

$$
\begin{aligned}
m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0\\
m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0\\
m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 - k_4x_4 &= 0\\
\ldots\ldots &= \ldots\ldots\ldots \qquad \ldots\ldots\ldots\\
m_n\ddot{x}_n - k_nx_{n-1} + k_nx_n &= 0
\end{aligned}
\tag{1}
$$

The above equations of motion which can also be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 & .. & 0 \\ 0 & m_2 & 0 & .. & 0 \\ & & m_3 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ .. \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & .. & 0 \\ -k_2 & k_2+k_3 & -k_3 & .. & 0 \\ 0 & -k_3 & k_3+k_4 & .. & 0 \\ .. & .. & .. & .. & 0 \\ 0 & 0 & 0 & .. & k_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (2)$$

The equation (2) then can be expressed in matrix form as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad (3)$$

where

$[M]$ is a square matrix of n^{th} order and with only diagonal elements in this case;
$[K]$ is a square stiffness matrix also of n^{th} order, this matrix being a symmetrical one; and
$\{x\}$ is a column matrix of *n* elements corresponding to the dynamic displacements of the respective *n* masses.

2.2 Characteristic Equation

If equation (3) is premultiplied by $[M]^{-1}$, it becomes

$$[I]\{\ddot{x}\} + [D]\{x\} = \{0\} \quad (4)$$

where $[D] = [M]^{-1}[K]$, is called a dynamic matrix.

Assuming harmonic motion $\ddot{x} = -\lambda\, X$, where $\lambda = \omega^2$, equation (4) becomes

$$[D - \lambda I]\{X\} = \{0\} \quad (5)$$

The solution of equation (5) is given as

$$|D - \lambda I| = 0 \quad (6)$$

which is the *characteristic equation* of the system. The roots λ_i of the characteristic equation are called *eigen values* and the natural frequencies ω_i of the system are determined from the by the relationship

$$\omega_i = \sqrt{\lambda_i} \quad (7)$$

If $m_1 = m_2 = m_3 = \cdots \ldots \ldots \ldots .. = m_n = m$, and $k_1 = k_2 = k_3 = \cdots \ldots \ldots \ldots .. = k_n = k$, dynamic matrix becomes

$$[D] = [M]^{-1}[K] = \begin{bmatrix} \frac{1}{m} & 0 & 0 & .. & 0 \\ 0 & \frac{1}{m} & 0 & .. & 0 \\ & & \frac{1}{m} & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 2k & -k & 0 & .. & 0 \\ -k & 2k & -k & .. & 0 \\ 0 & -k & 2k & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & k \end{bmatrix} \quad (8)$$

Characteristic equation of the system is then given by

$$\lambda^n + C_{n-1}\left(\frac{k}{m}\right)\lambda^{n-1} + C_{n-2}\left(\frac{k}{m}\right)^2 \lambda^{n-2} + C_{n-3}\left(\frac{k}{m}\right)^3 \lambda^{n-3} + \cdots \ldots \ldots \ldots \ldots \ldots$$

$$+C_2\left(\frac{k}{m}\right)^{n-2}\lambda^2 + C_1\left(\frac{k}{m}\right)^{n-1}\lambda + C_0\left(\frac{k}{m}\right)^n = 0 \tag{9}$$

A computer program is developed to determine the coefficients of characteristics equation and the obtained values of coefficients of equation (9) for the first 10 degree of freedom are tabulated in Table 1.

Table 1: Values of coefficients of characteristic equation

DOF (n)	**Values of Coefficients**										
	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
1	-1	1									
2	1	-3	1								
3	-1	6	-5	1							
4	1	-10	15	-7	1						
5	-1	15	-35	28	-9	1					
6	1	-21	70	-84	45	-11	1				
7	-1	28	-126	210	-165	66	-13	1			
8	1	-36	210	-462	495	-286	91	-15	1		
9	-1	45	-330	924	-1287	1001	-455	120	-17	1	
10	1	-55	495	-1716	3003	-3003	1820	-680	153	-19	1

3. Development of Polynomials to Determine the Coefficients Directly

By observing the coefficients, polynomials for each order of λ are developed as follows.

Constant Term

$$C_0(n) = (-1)^n$$

Coefficient of λ

$$C_1(n) = (-1)^{n+1}\,\frac{n(n+1)}{2}$$

Coefficient of λ^2

$$C_2(n) = (-1)^n\,\frac{n(n^2-1)(n+2)}{24}, \quad n \geq 2$$

Coefficient of λ^3

$$C_3(n) = (-1)^{n+1}\,\frac{n(n^2-1)(n^2-4)(n+3)}{72}, \quad n \geq 3$$

Coefficient of λ^4

$$C_4(n) = (-1)^n \frac{n(n^2-1)(n^2-4)(n^2-9)(n+4)}{40320}, \quad n \geq 4$$

Coefficient of λ^5

$$C_5(n) = (-1)^{n+1} \frac{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n+5)}{362880}, \quad n \geq 5$$

By comparing all the above polynomials a generalized polynomial for the coefficient of j^{th} order of λ (except C_0) for n degree of freedom system can be developed as

$$C_j(n) = (-1)^{n+j} \frac{n(n+j)}{(2j)!} \prod_{i=1,2,3,\dots}^{j-1} (n^2 - i^2) \tag{10}$$

4. Examples for Direct Method

Polynomial given by equation (10) is verified with the examples given below.

Example 1: 4 DOF system

Coefficients of characteristic equation for a system with four degree of freedom ($n = 4$) can be determined by using equation (10) as

$$C_0(4) = (-1)^4 = 1$$
$$C_1(4) = (-1)^5 \frac{4.5}{2!} = -10$$
$$C_2(4) = (-1)^6 \frac{4.6}{4!} \times (16-1) = 15$$
$$C_3(4) = (-1)^7 \frac{4.7}{6!} \times (16-1) \times (16-4) = -7$$
$$C_4(4) = (-1)^8 \frac{4.8}{8!} \times (16-1) \times (16-4) \times (16-9) = 1$$

Substituting these coefficients into equation (9), the characteristic equation for a system with four degree of freedom ($n = 4$) can be expressed as

$$\lambda^4 - 7\left(\frac{k}{m}\right)\lambda^3 + 15\left(\frac{k}{m}\right)^2 \lambda^2 - 10\left(\frac{k}{m}\right)^3 \lambda + \left(\frac{k}{m}\right)^4 = 0$$

Example 2: 7 DOF system

Coefficients of characteristic equation for a system with four degree of freedom ($n = 7$) can be determined by using equation (10) as

$$C_0(7) = (-1)^7 = -1$$
$$C_1(7) = (-1)^8 \frac{7.8}{2!} = 28$$
$$C_2(7) = (-1)^9 \frac{7.9}{4!} \times (49-1) = -126$$

$$C_3(7) = (-1)^{10}\frac{7.10}{6!} \times (49-1) \times (49-4) = 210$$

$$C_4(7) = (-1)^{11}\frac{7.11}{8!} \times (49-1) \times (49-4) \times (49-9) = -165$$

$$C_5(7) = (-1)^{12}\frac{7.12}{10!} \times (49-1) \times (49-4) \times (49-9) \times (49-16) = 66$$

$$C_6(7) = (-1)^{13}\frac{7.13}{12!} \times (49-1) \times (49-4) \times (49-9) \times (49-16) \times (49-25) = -13$$

$$C_7(7) = (-1)^{14}\frac{7.14}{14!} \times (49-1) \times (49-4) \times (49-9) \times (49-16) \times (49-25) \times (49-36) = 1$$

Substituting these coefficients into equation (9), the characteristic equation for a system with four degree of freedom ($n = 7$) can be expressed as

$$\lambda^7 - 13\left(\frac{k}{m}\right)\lambda^6 + 66\left(\frac{k}{m}\right)^2\lambda^5 - 165\left(\frac{k}{m}\right)^3\lambda^4 + 210\left(\frac{k}{m}\right)^4\lambda^3 - 126\left(\frac{k}{m}\right)^5\lambda^2 + 28\left(\frac{k}{m}\right)^6\lambda - \left(\frac{k}{m}\right)^7 = 0$$

5. Conclusion

In this paper a generalized polynomial expression is developed which can be used to determine the coefficients of the characteristic equation of any MDOF system which is otherwise done through the expansion of the determinant $|D - \lambda I|$. Examples for 4 DOF and 7 DOF systems are also presented to verify the method.

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