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A Note on Full-Rank Factorization of Matrix

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Abstract: We exhibit that the Singular Value Decomposition of a matrix A_{nxm} implies a natural full-rank factorization of the matrix A.

Keywords: Full-rank factorization, singular value decomposition, factorization of matrices

1. Introduction

Let's consider a matrix A_{nxm} such that rank $A = p$, then its full-rank factorization means the existence of matrices F_{nxp} and G_{pxm} with the properties [1]:

$$A = F\,G, \qquad \text{rank}\,F = \text{rank}\,G = p\,; \tag{1}$$

then in Section 2, we show that the Singular Value Decomposition (SVD) [3, 4, 6, 7, 8, 9, 10, 12] gives a natural full-rank factorization of A:

$$A_{nxm} = U_{nxp}\,W_{pxm}\,, \qquad \text{Col}\,A = \text{Col}\,U \qquad \& \qquad \text{Row}\,A = \text{Row}\,W. \tag{2}$$

2. SVD and Full-rank Factorization

For any real matrix A_{nxm}, Lanczos [9, 10] introduces the Jordan matrix [5, 11, 13]:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \tag{3}$$

and he studies the eigenvalue problem:

$$S\vec{\omega} = \lambda\vec{\omega}, \tag{4}$$

where the proper values are real because S is a real symmetric matrix. Besides:

$$\text{rank}\,A \equiv p = \text{Number of positive eigenvalues of } S, \tag{5}$$

such that $1 \leq p \leq \min(n, m)$. Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, \dots, \lambda_p, -\lambda_1, -\lambda_2, \dots, -\lambda_p, 0, 0, \dots, 0, \tag{6}$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of *S*, named 'essential axes' by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \begin{matrix} n \\ m \end{matrix}, \tag{7}$$

then (3) and (4) imply the Modified Eigenvalue Problem:

$$A_{nxm}\, \vec{v}_{mx1} = \lambda\, \vec{u}_{nx1}, \qquad A^T{}_{mxn}\, \vec{u}_{nx1} = \lambda\, \vec{v}_{mx1}, \tag{8}$$

hence:

$$A^T A \vec{v} = \lambda^2 \vec{v}, \quad A A^T \vec{u} = \lambda^2 \vec{u}, \tag{9}$$

with special interest in the associated independent vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = (\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p), \quad V_{mxp} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p), \tag{10}$$

verifying $U^T U = V^T V = I_{pxp}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk}, \tag{11}$$

therefore $\vec{\omega}_j \cdot \vec{\omega}_k = 2\delta_{jk}$, $j, k = 1, 2, \dots, p$. Thus, the SVD express [7, 8, 9, 10, 14] that *A* is the product of three matrices:

$$A_{nxm} = U_{nxp}\, \Lambda_{pxp}\, V^T{}_{pxm}, \quad \Lambda = \text{Diag}\,(\lambda_1, \lambda_2, \dots, \lambda_p). \tag{12}$$

This relation tells that in the construction of *A* we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to *A*.

The expression (12) is a natural full-rank factorization of *A* because it has the structure (2) with U_{nxp} given by (10) and:

$$W_{pxm} = \Lambda_{pxp}\, V^T{}_{pxm} = \begin{pmatrix} \lambda_1\, \vec{v}_1{}^T \\ \vdots \\ \lambda_p\, \vec{v}_p{}^T \end{pmatrix}, \tag{13}$$

then it is clear that Col U = Col A & Row W = Row V^T = Col V = Row A, in according with (2), therefore the columns of A are all linear combinations of the columns of U, and the rows of A are all linear combinations of the rows of W. In [2], there is the factorization of (1) for the particular case $n = m$ and $p < n$, that is, for singular square matrices.

3. Conclusion

Our approach shows that the Singular Value Decomposition gives a natural full-rank factorization for an arbitrary matrix, which is useful to determine the Moore-Penrose pseudoinverse [1, 10, 14].

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