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Mysteries of Eigenvalues, Eigenvectors & their Applications in the Diagonalization of a Matrix & in the Cayley-Hamilton Theorem to Find the Matrix Inverse

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Abstract: In this paper we deal with eigenvalues and eigenvectors (E-values & E-vectors) in diagonalizating a square matrix and in the Cayley-Hamilton theorem used to find the inverse of a given square matrix.

1. Introduction

If A $[a_{ij}]_{n\times n}$ = $\begin{bmatrix} a_{11} & a_{12}\dots & a_{1j}\dots & \dots & a_{1n} \\ a_{21} & a_{22}\dots & a_{2j}\dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2}\dots & a_{ij}\dots & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2}\dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$ is a square matrix of order n then the matrix

(A-λI) i.e.

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1j}\dots & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda\dots & a_{2j}\dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij}-\lambda\dots & \dots & a_{in} \\ \dots & & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn}-\lambda \end{bmatrix}$$

is called the characteristic matrix of A. If the determinant of this characteristic matrix of A is taken and equated to zero i.e.

$$|A-\lambda I| = 0 \qquad \text{(i)}$$

i.e.

$$\begin{vmatrix} a_{11}-\lambda & a_{12}\cdots & a_{1j}\cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda\ldots & a_{2j}\cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2}\cdots & a_{ij}-\lambda\ldots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2}\cdots & a_{nj} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

i.e. $\lambda^n + p_1\lambda^{n-1} + \ldots + p_n = 0$ (ii)

it is called the characteristic equation of A. This equation naturally has n roots, so these n-roots of λ are called the characteristic roots, latent roots, or eigenvalues of A.

Again, if $A = [a_{ij}]_{n\times n}$ is a square matrix of order n and $X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ is a column vector which is transformed by A into its scalar multiple such that

$AX = \lambda X$ (iii)

And if I is the unit matrix of the same order then (iii) can be written as

$AX = \lambda IX$

or $(A - \lambda I)\, X = 0$ (iv)

which is called the characteristic matrix equation and this equation (iv) represents n homogeneous equations:

$$(a_{11} - \lambda)\, x_1 + a_{12}x_2 + \ldots + a_{1n}\, x_n = 0$$
$$a_{21}\, x_1 + (a_{22} - \lambda)\, x_2 + \ldots + a_{2n}\, x_n = 0$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$a_{n1}\, x_1 + a_{n2}x_2 + \ldots + (a_{nn} - \lambda)\, x_n = 0$$

This system of n homogeneous linear equations will have a non-zero solution if $|A - \lambda I|$ is singular i.e. $|A - \lambda I| = 0$. Moreover the roots of $(A-\lambda I) = 0$ gives the n-eigenvalues ($\lambda_1 \ldots \lambda_n$). To each eigenvalue of A, there corresponds a non-zero solution to the vector $X = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$.

This vector X is called the eigenvector of A (corresponding to that particular eigenvalue of A).

Note that the eigenvector corresponding to each eigenvalue is not unique. So any scalar multiple of the E-vector are also the E-vectors of A corresponding to those particular E-values of A.

For example, if $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is a square matrix then the characteristic equation is

$$|A - \lambda I| = 0$$

i.e. $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1 \text{ or } 3$

which are the E-values of A. Again, if $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is an E-vector then the corresponding matrix equation is

$$(A - \lambda I)\, x = 0 \qquad \text{(v)}$$

$$\text{i.e, } \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(vi)}$$

When $\lambda = 1$ then (vi) reduces to

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} x + y = 0 \\ \text{and } x + y = 0 \end{matrix}\right\}$$

(No of LI solutions = unknowns-rank = 2-1=1)

If we assign x = 1 then we get y = -1 so the E-vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the set of all E-vectors is $\begin{bmatrix} k_1 \\ -k_1 \end{bmatrix}' (k_1 \neq 0)$.. Similarly when $\lambda = 3$ then (vi) give the E-vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the set of all eigenvectors is $\begin{bmatrix} k_2 \\ k_2 \end{bmatrix}' (k_2 \neq 0)$.

2. Application of E-values and E-vectors in the Diagonalization of a Square Matrix

Now we shed light on the reduction of a given square matrix into the diagonal form through the transforming matrix P to the diagonal from D.

It is clear that "If a square matrix $A = [a_{ij}]_{n \times n}$ of order n has n linearly independent E-vectors then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix (whose diagonal elements are the same E-values)". For example, if we are given a matrix

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ then a transforming matrix P can be found such that $P^{-1}AP \left(= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} (= D) \right)$

which transforms A into the diagonal form $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ whose diagonal elements are the same E-values.

(Note that the sum of all diagonal elements of a given square matrix A is equal to the sum of all E-values of A).

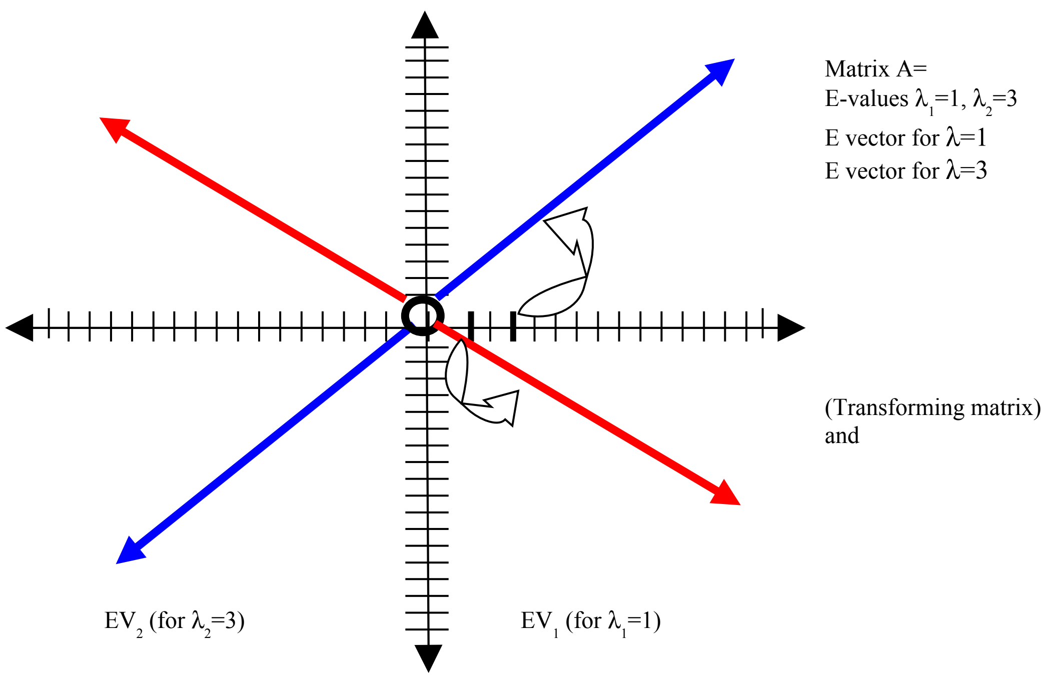
Moreover, if P = $[X_1\ X_2]$ be a column vector of the transforming matrix P of a given square matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The E-values of A are 1 and 3 and the E-vectors corresponding to λ=1 and λ=3 are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $(=X_1)$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(=X_2)$ respectively.

Then we make a new matrix P = $[X_1\ X_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

which is the transforming matrix which transforms A into the diagonal from (D) i.e.

$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ (= D) which is the diagonal from D and the diagonal elements are the same E-values. This is the direct relation among the E-values, E-vectors, transforming matrix P and the diagonal form D of a given square matrix A which has a relation like a grandfather.

This relationship can be sketched/represented graphically as shown below.



This pictorial representation shows the deep relationship, among the E-values and the E-vectors of a given square matrix A.

3. Application of Cayley-Hamilton theorem in finding the inverse of a given square matrix

Here we show the application of the Cayley-Hamilton Theorem (C-H theorem) to find the inverse of a given square matrix in simpler form. We know that the Cayley-Hamilton theorem states that "every square matrix A satisfies its own characteristic equation".

For, if $A = [a_{ij}]_{n\times n}$ is a square matrix of order n then its characteristic equation is

$$|A - \lambda I| = 0$$

i.e. $$\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-1} + \ldots + p_n = 0 \quad \text{(vii)}$$

then by C-H theorem $A^n + p_1A^{n-1} + p_2A^{n-1} + \ldots + p_n I = 0$ (viii)

where I is the unit matrix of the same order. The inverse of the given square matrix A can be obtained directly by using this Cayley-Hamilton theorem. From the equation (viii)

$$A^n + p_1A^{n-1} + p_2A^{n-1} + \ldots + p_n I = 0$$

Now multiplying both sides by A^{-1}

we get $A^{-1} (A^n + p_1A^{n-1} + \ldots + p_n I = A^{-1}.0$

or $A^{n-1} + p_1 A^{n-2} + \ldots + p_{n-1}I + A^{-1} . p_n = 0$

$$\Rightarrow \quad A^{-1} = -\frac{1}{p^n}\left[A^{n-1} + p_1 . A^{n-2} + \ldots + p_{n-1} I\right]$$

results in the inverse of a square matrix A. This is a very nice application of the Cayley-Hamilton theorem from which we get A^{-1} by a short and sweet method without using the lengthy formula $A^{-1} = \frac{Adj\ .A}{|A|} (A \neq 0)$

which are nice applications of E-values & E-vectors in the diagonalization of a matrix & in the Cayley-Hamilton theorem.

Some common properties of E-values, E-vectors, transforming matrix & diagonal form (matrix)

1. Any square matrix A and its transpose have the same E-values.
2. The trace of the matrix equals to the sum of E-values of the same matrix.

   i.e. $(a_{11} + a_{22} + \ldots + a_{nn} = \lambda_1 + \lambda_2 + \ldots + \lambda_n)$

3. The determinant of the matrix A equals the product of E-values of A.
4. If ($\lambda_1, \lambda_2 \ldots \lambda_n$) are the n E-values of A then the E-values of kA are ($k\lambda_1, k\lambda_2, \ldots k\lambda_n$).
5. E-values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \cdots \frac{1}{\lambda_n}$.
6. If λ is an E-value of a matrix A then $\frac{1}{\lambda}$ is an E-value of A^{-1}.
7. If ($\lambda_1, \lambda_2 \ldots \lambda_n$) are the E-values of a square matrix A, then A^m has the E-values $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$, where m is a positive integer.
8. Zero is an E-value of a square matrix A if and only if A is singular i.e. $|A| = 0$.
9. The E-values of a triangular matrix are just the diagonals elements of the matrix.
10. Two matrices A & $P^{-1}AP$ have the same E-values.

Eigen values of some special type of matrices

1. The E-values of the Hermitian matrix are real.
2. The E-values of a real symmetric matrix are all real.
3. Every E-value of a skew-Hermitian matrix is either 0 or a pure imaginary number.
4. The E-values of a unitary matrix are of unit modulus.
5. The E-values of an orthogonal matrix are of unit modulus.
6. If λ is an E-value of an orthogonal matrix A then $\frac{1}{\lambda}$ is also an E-value of A.

4. Conclusion

E-values, E-vectors have a nice application in the diagonalization of a square matrix and in the Cayley-Hamilton theorem used to find the matrix inverse and in all the endeavors of medical, engineering and social sciences.

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