

Polynomials and Taylor's Approximations

Nhuchhe Shova Tuladhar

Department of Science and Humanities, Pulchowk Campus, Institute of Engineering
Tribhuvan University, Kathmandu, Nepal
Corresponding author: nhuchhet@gmail.com

Received: Dec. 5, 2015 Revised: April 15, 2016 Accepted: Aug. 25, 2016

Abstracts: The main objective of this article is to make a formal description of the polynomial, polynomial equations with definitions and their properties. Besides studying some of its uses in real life situations, we shall discuss polynomial approximation using higher order derivatives.

Key words: Polynomials, Polynomial Equations, Taylor's Approximation.

1. Introduction

In mathematics, a polynomial is the simplest class of mathematical expression constructed from variables (called indeterminate) and constants using the operations of addition, subtraction, multiplication and non-negative integer exponents, [10, 12]. Polynomial comes from the Greek *poly*, "many" and Latin *binomium*, "binomial". Polynomials appear in a wide variety of areas of mathematics and science. For example, they are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; they are used to define **polynomial functions**, which appear in settings ranging from basic chemistry and physics to economics and social science; they are used in calculus and numerical analysis to approximate other functions. In advanced mathematics, polynomials are used to construct polynomial rings and algebraic varieties, central concepts in algebra and algebraic geometry [3, 6, 7].

2. Polynomial Functions

A function F of one argument is called a polynomial function if it satisfies

$$F(x) = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_1 x + A_0,$$

for all arguments x , where n is a non-negative integer and the coefficients $A_0, A_1, A_2, \dots, A_n$ are constants and $A_n \neq 0$. Polynomials of degree zero, one, two, three, four and five are respectively called *constant polynomials*, *linear polynomials*, *quadratic polynomials*, *cubic polynomials*, *quadratic polynomials* and *quintic polynomials*. Polynomial functions are continuous, smooth, entire, computable, etc. A one-term polynomial is called a *monomial*; a two-term polynomial is

called a *binomial*, and so on. A polynomial in one variable is called a *univariate*, a polynomial in more than one variables is called a *multivariate polynomial*.

All polynomials with coefficients in a unique factorization domain is written as a product of irreducible polynomials and a constant. This factored form is unique up to the order of the factors. The graphs of polynomial functions are continuous and have no sharp corners. The sign of the leading coefficient determines the end behavior of the function. The degree n determines the number of complex zeros of the function. The number of real zeros of the function will be less than or equal to the number of complex zeros. The real zeros of a polynomial function may be found by factoring (where possible) or by finding the intersection with the x -axis. The number of times a zero occurs is called its multiplicity. If a function has a zero of odd multiplicity, the graph of the function crosses the x -axis at that x -value. However, if a function has a zero of even multiplicity, the graph of the function only touches the x -axis at that x -value.

Formulae for expressing the roots of polynomials of degree 2 in terms of square roots have been known since ancient times. Several workers like Niccolo Fontana Tartaglia, Lodovico Ferrari, Gerolamo Cardano, Vieta etc have made their significant contributions in the 16th century to develop formulae for the cubic and quartic polynomials. In 1824, Niels Henrik Abel proved the striking result involving only arithmetic operations and radicals that expresses the roots of a polynomial of degree 5 or greater in terms of its coefficients (see Abel-Ruffini theorem). In 1830, Évariste Galois, studied the permutations of the roots of a polynomial, extended the Abel-Ruffini theorem. This result marked the establishment of Galois Theory and Group theory, two important branches of modern mathematics [2, 3, 4, 10, 12].

3. Fundamental Properties of Polynomials

The complex structure of polynomial functions makes them useful in analyzing using polynomial approximations. In calculus, Taylor's theorem states that every differentiable function locally looks like a polynomial function, and the Stone-Weierstrass theorem, which states that every continuous function defined on a compact interval of the real axis can be approximated on the whole interval as closely as desired by a polynomial function.

Based on the notion of zeros of a polynomial, we get the following properties of rational integral domain, [3, 8, 10, 12].

- i. The set $D[x]$ of all polynomials in x over an integral domain D is an integral domain.
- ii. (*Continuity of Polynomial Function*):

Every polynomial function $F: R \rightarrow R$ defined by

$$F(x) = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_1 x + A_0,$$

where A_1, A_2, \dots, A_n being real, is continuous on R .

- iii. If F and G are polynomials and $G \neq 0$, then the rational function $h: R \rightarrow R$ defined by $h =$

$\frac{F}{G}$ is continuous on R except at points for which $G = 0$.

iv. *Division Algorithm:*

For any non-zero polynomials $F(x)$ and $G(x)$ over a field K and $G(x) \neq 0$, there exist unique polynomials $S(x)$ and $R(x)$ such that

$$F(x) = G(x)S(x) + R(x),$$

where $R(x)$ is zero or of degree less than that of $G(x)$.

v. *Remainder Theorem:*

The value of polynomial $F(x)$ at $x = c$ equals the remainder obtained on dividing $F(x)$ by $x - c$.

In other words, if a polynomial $F(x)$ is divided by a linear polynomial $x - c$, then the remainder is $F(c)$.

vi. *Factor Theorem:*

The linear polynomial $x - c$ is a factor of a polynomial $F(x)$ if and only if $F(c)$ is zero.

In other words, if a polynomial $F(x)$ is divided by the linear polynomial $x - c$ and remainder $R(c) = 0$, then $x - c$ is a factor of $F(x)$.

vii. *Fundamental Theorem of Algebra:*

Every polynomial with complex coefficients (over a field of complex numbers) of degree ≥ 1 , has at least one root in \mathbb{C} . In fact, a polynomial $F(x)$ over a field of complex numbers \mathbb{C} , of degree n has exactly n zeros in \mathbb{C} .

If the leading coefficient a_n in a polynomial of degree n , then there exist complex numbers c_1, c_2, \dots, c_n such that

$$F(x) = A_n(x - c_1)(x - c_2) \dots (x - c_n).$$

viii. *Rolle's Theorem:*

If $F(x)$ is a polynomial over the field of real numbers \mathbb{R} , and if a and b are real number with $F(a)$ and $F(b)$ having opposite signs, one positive and other negative, then $F(x)$ has a real zero between a and b .

Polynomial equations are used in a wide variety of areas of mathematics and science. It appears in basic chemistry, physics, economics and social sciences. It is used in calculus and numerical analysis to approximate other functions. In advanced mathematics, polynomials are used to construct polynomial rings, a central concept in algebra and algebraic geometry. Polynomials are frequently used to encode information about some other object. The characteristic polynomial of a matrix or linear operator contains information about the operator's eigenvalues. The chromatic polynomial of a graph counts the number of proper colorings of that graph.

4. Polynomial Equations

A *polynomial equation*, also called an algebraic equation is of the form

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_1 x + A_0 = 0.$$

A polynomial identity is a polynomial equation for which all possible values of the unknown are the solutions.

In elementary algebra, quadratic formula are given for solving all second degree polynomial equations in one variable. There are also formulae for the cubic and quartic equations. For higher degrees, Abel–Ruffini theorem asserts that there can not exist a general formula, only numerical approximations of the roots may be computed. The number of solutions may not exceed the degree when the complex solutions are counted with their multiplicity. This fact is called the fundamental theorem of algebra.

For a set of polynomial equations in several unknowns, there are algorithms to decide if they have a finite number of complex solutions. It has been shown by Richard Birkeland and Karl Meyr that the roots of any polynomial may be expressed in terms of multivariate hypergeometric functions. Ferdinand von Lindemann and Hiroshi Umemura showed that the roots may also be expressed in terms of Siegel modular functions, generalizations of the theta functions that appear in the theory of elliptic functions. We now state some of the fundamental properties of the polynomial equations, [2, 3, 4, 10,12].

- i. *Every equation of degree n has exactly n roots.*
- ii. *In every equation with real coefficients, imaginary roots occur in conjugate pairs.*
In other words, if $a + ib$ is one of the roots, then $a - ib$ is other root.
- iii. *In every equation surd roots occur in conjugate pairs.* From this result, it follows that if $a + \sqrt{b}$ is one of the roots, then $a - \sqrt{b}$ is also a root.
- iv. *For any two real numbers a and b , if $F(a)$ and $F(b)$ have opposite signs, then the equation $F(x) = 0$ has at least one root between a and b .*
Consider, $F(x) = x^2 - x - 2 = 0$, then $F(0) = -2 < 0$ and $F(3) = 4 > 0$.
So $F(0)$ and $F(3)$ have opposite signs. Hence it has at least one root between 0 and 3.
A simple calculation shows that there is a root 2 between 0 and 3.
- v. *Every equation of an odd degree has at least one real root whose sign is opposite to that of its absolute term.*
Consider $F(x) = x^3 - 4x^2 + x + 6 = 0$. Then $F(-2) = -20 < 0$ and $F(0) = 6 > 0$
i.e. $F(-2)$ and $F(0)$ have opposite signs. So there should be at least one root between -2 and 0.
Moreover absolute term is positive, so one root must be negative. A simple calculation shows that there is a negative root -1 among the roots $-1, 2$ and 3 .
- vi. *Every equation of an even degree, whose absolute term is negative has at least two real roots, one positive and one negative.*
Consider an equation $F(x) = x^2 + x - 12 = 0$, which is of even degree with negative absolute term.
It has two real roots one positive and other negative namely $x = -3$ and $x = 4$.

Similarly, for $F(x) = x^4 + 2x^3 - 4x^2 - 5x - 6 = 0$, which is an even degree with negative absolute term.

It has two real roots -2 and 3 and other two imaginary roots.

vii. An equation $F(x) = 0$ cannot have more positive roots than the number of changes of sign in $F(x)$

(from $+$ to $-$ or from $-$ to $+$) and cannot have more negative roots than there are changes of sign in $F(-x)$.

This rule is known as *Descartes' Rules of Signs* and helps us to determine the nature of some of the roots without actually determining them i.e., without solving the equation.

Consider $F(x) = x^3 - 2x^2 - 5x + 6 = 0$. Here, the signs of the terms of $F(x)$ are

$$+ \quad - \quad - \quad +.$$

So the number of changes in signs in $F(x) = 2$ and hence the number of positive roots cannot be greater than 2. Replacing x by $-x$, then

$$F(-x) = -x^3 - 2x^2 + 5x + 6.$$

So that the signs of the terms are $- \quad - \quad + \quad +$. Hence the number of changes of sign in $F(-x)$ is 1.

So there must be only one negative root of $F(x) = 0$. In fact, there are three roots namely $1, -2$ and 3 .

5. Approximation of Polynomials

Many functions like $e^{x^2}, \sin x, \cos(e^{x^2})$ etc are much more difficult to work with than linear polynomial of the form $mx + b$, and so many times it is useful to approximate such complicated expressions by a linear function of the form $f(x) = mx + b$. Because of simplicity in form and applicability of well-known algebraical and analytical operational rules, polynomials are often used for such purposes, [1, 9, 11, 13]. For a small interval of x values, differential calculus focuses on the construction and use of tangent lines at various values of x . By using higher derivatives, the idea of a tangent line can be extended to the idea of polynomials of higher degree which are "tangent" in some sense to a given curve. This is accomplished by using a polynomial of high degree, and or narrowing the domain over which the polynomial has to approximate the function, see [5, 11, 13,14]. Narrowing the domain can often be done through the use of various addition or scaling formulas for the function being approximated.

Using the notion of a derivative, we approximate a real valued function p differentiable in an open interval (a, b) and continuous in the closed interval $[a, b]$, by a linear function or first-degree Taylor's polynomial F_1 defined by

$$F_1(x) = p(r) + p'(r)(x - r), r \in [a, b]$$

for x 'close' to r .

The graph of this approximating polynomial for p is the tangent line

$$y = p(r) + p'(r)(x - r) \text{ at } x = r.$$

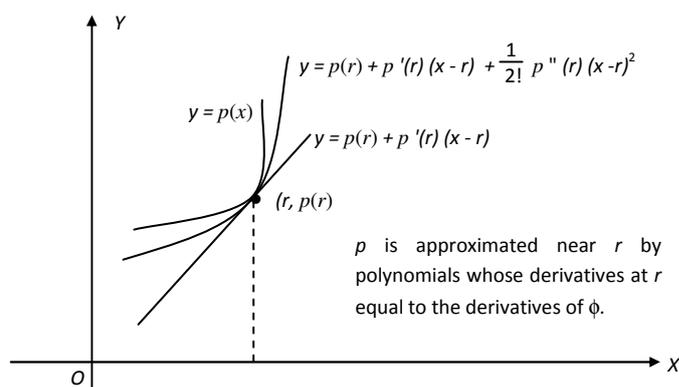


Fig. 1: Approximating polynomial for p

For a function p which is twice differentiable in (a, b) and whose first derivative is continuous in $[a, b]$, then

its *Taylor's* second approximation $F_2(x)$ can be obtained in the form

$$F_2(x) = p(r) + p'(r)(x - r) + \frac{1}{2!} p''(r)(x - r)^2, \quad r \in [a, b]$$

for x 'close' to r .

Interpreted geometrically, it means

- i. $F_2(x) = p(r)$, i.e., $F_2(x)$ has the same value at r as p .
- ii. $F_2'(x) = p'(r) + p''(r)(x - r)$, i.e., $F_2'(r) = p'(r)$
i.e., the slope of the tangent line to F_2 at r is the same as the slope of the tangent line to p at r .
- iii. $F_2''(x) = p''(r)$, $F_2''(r) = p''(r)$
i.e., the second derivative of F_2 at x is the same as that of p at r .
- iv. If we further assume that p is 3-times differentiable in the open interval (a, b) and p has continuous second derivative in the closed interval $[a, b]$, we shall have an *Taylor's* approximation $F_3(x)$ of p in the form

$$F_3(x) = p(r) + p'(r)(x - r) + \frac{1}{2!} p''(r)(x - r)^2 + \frac{1}{3!} p'''(r)(x - r)^3,$$

for x 'close' to r .

As an illustration, to find the third degree Taylor's polynomial approximation for $p(x) = \sqrt{x}$ at point $x = 1$, we have

$$p'(x) = \frac{1}{2\sqrt{x}}, \quad p''(x) = -\frac{1}{4x^{3/2}} \quad \text{and} \quad p'''(x) = \frac{3}{8x^{5/2}}$$

and so

$$p(1) = 1, \quad p'(1) = \frac{1}{2}, \quad p''(1) = -\frac{1}{4} \quad \text{and} \quad p'''(1) = \frac{3}{8}.$$

So the Taylor's polynomial of degree 3 for the function $p(x)$ at $x = 1$ yields

$$F_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

Similarly at $x = 0$, since $p'(0)$, $p''(0)$ and $p'''(0)$ do not exist. Therefore, Taylor's polynomial at point $x = 0$ does not exist.

In general, if a real valued function p having continuous $(n-1)^{\text{st}}$ derivative in the open interval (a, b) has a finite n^{th} derivative in the closed interval $[a, b]$, it can be approximated more accurately by a polynomial of degree n in the form

$$F_n(x) = p(r) + p'(r)(x-r) + \frac{1}{2}p''(r)(x-r)^2 + \dots + \frac{1}{n!}p^{(n)}(r)(x-r)^n$$

for x 'close' to r .

The polynomial F_n is called a *Taylor polynomial of degree n* for p at r . An important but obvious property of Taylor polynomial is

$$F_n^{(k)}(r) = p^{(k)}(r), \quad \text{for } k = 0, 1, 2, \dots, n.$$

In particular, to find the Taylor's polynomial of degree n for the function $p(x) = e^x$ at point $x = 0$,

$p^{(n)}(x) = e^x$ for every $n \in \mathbf{Z}^+$ and therefore

$$p'(0) = p''(0) = \dots = p^{(n)}(0) = 1.$$

Taylor's polynomial of degree n for e^x at point $x = 0$ is

$$\begin{aligned} F_n(0) &= p(0) + p'(0)x + \frac{1}{2}p''(0)x^2 + \dots + \frac{1}{n!}p^{(n)}(0)x^n \\ &= 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}. \end{aligned}$$

6. Conclusion

In mathematics, approximation theory is concerned with how functions can best be approximated as close as possible with simpler functions. An approximation of more complicated functions by polynomials is a basic building block for a numerical technique. Sometimes, approximation of functions by generalized Fourier series is based upon summation of a series of terms based upon orthogonal polynomials.

References

- [1] Abramowitz M and Stegun IA (1962), *Handbook of Mathematical Functions*. National Bureau of Standards, Washington.
- [2] Atkinson KE (1989), *An Introduction to Numerical Analysis*, 2nd Ed. Wiley, New York.
- [3] Bhattarai HN and Dhakal GP (2064), *Undergraduate Algebra*, First Edition, Vidyarthi Pustak Bhandar, Kathmandu.
- [4] Eves HW (1982), *An Introduction to the History of Mathematics* (5th ed.), Saunders College Publishing, USA.
- [5] Gupta SL and Nisha R (1993): *Fundamentals Real Analysis* (3rd ed.), Vikas Publishing House (P.) Ltd., New Delhi.
- [6] Hungerford TW (1974), *Algebra, Graduate Texts in Mathematics*, Spring-Verlag, New York.
- [7] Kam-Timleung (1972), *Linear Algebra and Geometry*, Hong Kong University Press.
- [8] Kumaresan S (2009), *Linear Algebra, A Geometric Approach*, PHI Learning Pvt. Ltd., New Delhi, India.
- [9] Malik SC and Arora S (1999), *Mathematical Analysis*, (2nd ed.), New Age International Pvt. Ltd., India.
- [10] Prasad (1993), *A Text Book on Algebra and Theory of Equations*, Pothishala P. Limited India.
- [11] Rudin W (1964), *Principles of Mathematical Analysis*, MC Graw Hill Book Co., New York, USA.
- [12] Shrestha RM, Bajracharya S and Pahari NP (2015), *Elementary Linear Algebra*, Sukunda Pustak Bhawan, Kathmandu.
- [13] Shrestha RM and Pahari NP (2014), *Fundamentals of Mathematical Analysis*, Sukunda Pustak Bhawan, Kathmandu, Nepal.
- [14] Thomas Jr. GB, Finney RL and Weir MD (1999), *Calculus and Analytical Geometry*, (9th ed.), Addison Wesley Longman.