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Petersen-Varchenko's Identity for Stirling Numbers of the First Kind

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Abstract: Stirling numbers of the first kind has some interesting interpretations. In this short paper, we exhibit an elementary deduction of an identity for $S_n^{(m)}$ obtained by Petersen-Varchenko.

Keywords: Stirling numbers of the first kind, Polynomials

1. Introduction

Petersen-Varchenko [7] proved the following identity:

$$S_n^{(n-k)} = \frac{1}{k} \sum_{m=1}^{k} (-1)^m \binom{m+n-k}{m+1} S_n^{(m+n-k)}, \qquad 1 \leq k \leq n, \tag{1}$$

for the Stirling numbers of the first kind [9]. On the other hand, we know the expression [1, 3, 8]:

$$S_n^{(n-k)} = \sum_{j=n-k-1}^{n-1} (-1)^{n-k+j-1} \binom{j}{n-k-1} S_{n-1}^{(j)} \,. \tag{2}$$

In Section 2, we show that (2) implies (1).

2. Petersen-Varchenko's Formula

From (2), we have

$$S_n^{(n-k)} = -\sum_{m=-1}^{k-1} (-1)^m \binom{m+n-k}{m+1} S_{n-1}^{(m+n-k)}$$

$$= -\sum_{m=1}^{k-1} (-1)^m \binom{m+n-k}{m+1} S_{n-1}^{(m+n-k)} + S_{n-1}^{(n-k-1)} - (n-k)\, S_{n-1}^{(n-k)}, \tag{3}$$

and in the left side of (3) we employ the recurrence [8]

$$S_n^{(n-k)} = S_{n-1}^{(n-k-1)} - (n-1)\, S_{n-1}^{(n-k)},$$

thus

$$(k-1)\, S_{n-1}^{(n-k)} = \sum_{m=1}^{k-1} (-1)^m \binom{m+n-k}{m+1} S_{n-1}^{(m+n-k)} ,$$

where we make the changes $n \longrightarrow n+1$ and $k \longrightarrow k+1$ to obtain (1), QED.

Now we shall deduce an alternative relation to (1); in fact, from [2, 4] we have the expression:

$$S_n^{(n-k)} = \binom{n-1}{k} B_k^{(n)} , \tag{4}$$

with the participation of the Nörlund polynomials $B_m^{(z)}$ [5], that is,

$$S_n^{(j)} = \binom{k+j-1}{k} B_k^{(k+j)} . \tag{5}$$

Besides in [2, 5], we find the property:

$$(-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=0}^{k} \binom{k+j-1}{k} \binom{k-z}{k+j} \binom{k+z}{k-j} B_k^{(k+j)}, \tag{6}$$

$$\Rightarrow \quad B_k^{(n)} \overset{(5)}{=} \frac{(-1)^k}{\binom{k-n}{k}} \sum_{j=0}^{k} \binom{n}{k+j} \binom{2k-n}{k-j} S_n^{(j)} , \tag{7}$$

then from (4) and (7) we obtain the following identity for Stirling numbers of the first kind:

$$S_n^{(n-k)} = \sum_{j=1}^{k} \binom{n}{k+j} \binom{2k-n}{k-j} S_{k+j}^{(j)} , \qquad 1 \le k \le n , \tag{8}$$

which if $k = 1, 2$ gives the known results [3]:

$$S_n^{(n-1)} = -\binom{n}{2}, \qquad\qquad S_n^{(n-2)} = \frac{1}{4}\binom{n}{3}(3n-1). \tag{9}$$

It is possible to construct many identities for $S_n^{(m)}$, for example, from [3], we have:

$$(n-k)\, S_n^{(k)} = \sum_{j=k+1}^{n} (-1)^{k+j} \binom{j}{k-1} S_n^{(j)} , \tag{10}$$

and from [8]

$$\frac{(-1)^n}{n!} \sum_{j=k-1}^{n} (-1)^{k+j-1} \binom{j}{k-1} S_n^{(j)} = \sum_{r=0}^{n} \frac{(-1)^r}{r!} S_r^{(k-1)} , \tag{11}$$

we deduce the following relation

$$S_{n+1}^{(k)} = (-1)^n\, n! \sum_{m=0}^{n} \frac{(-1)^m}{m!} S_m^{(k-1)} . \tag{12}$$

3. Conclusion

Petersen-Varchenko [7] obtain formulas for the growth rate of the number of certain paths in a multi-dimensional analogue of the Eulerian graph, and as a corollary they deduce the identity (1). Our approach shows that (1) can be proved via an elementary process.

References

[1] Benjamin AT, Preston GO and Quinn JJ (2002), A Stirling encounter with harmonic numbers, *Maths. Mag.,* **75(2) :** 95-103.

[2] Carlitz L (1960), Note on Nörlund polynomials $B_n^{(z)}$, *Proc. Am. Math. Soc*., **11(3) :** 452-455.

[3] Comtet L (1974), *Advanced combinatorics,* D. Reidel Pub, Dordrecht, Holland.

[4] Gessel IM (2005), On Miki's identity for Bernoulli numbers, *J. Number Theory*, **110(1) :** 75-82.

[5] Gould HW (1960), Stirling number representation problems, *Proc. Am. Math. Soc.,* **11(3) :** 447-451.

[6] Nörlund NE (1924), *Vorlesungen über differenzenrechnung,* Springer-Verlag, Berlin.

[7] Petersen K, Varchenko A (2012), Path count asymptotics and Stirling numbers, *Proc. Am. Math. Soc.,* **140(6) :** 1909-1919.

[8] Quaintance J and Gould HW (2016), *Combinatorial identities for Stirling numbers,* World Scientific, Singapore.

[9] Riordan J (1968), *Combinatorial identities,* John Wiley & Sons, New York.