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Research Article

GENERALIZED SEQUENCE OF FUNCTIONS

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ABSTRACT

Uniform convergence is a more stringent type of convergence which preserves at least some shared properties of functions. In this sense, a unified method of generalized sequence of functions is more beneficial to study limiting operations. The purpose of this paper is to study the problem in a unifying approach using generalized sequence in order to obtain the basic result in general form and then indicate their application to concrete cases. It is quite observed that among various approaches, the abstract method is the simplest and most economical method for treating mathematical systems.

Keywords: Generalized sequence, pointwise convergence, uniform convergence

INTRODUCTION

In many research works, it is often necessary to build a sequence of functions which approximates a function under study. In doing so, one has to solve the following type of question (Rudin, 1976, Royden, 2001; Apostol, 2002):

"*If each term of a sequence of functions has a certain property such as continuity, differentiability and integrability to what extent is this property transformed to the limit function?*"

Answer to the above question separately, i.e., sequence and series of functions, integrals and improper integrals depending on parameters can be found on almost all textbooks in real analysis.

The main purpose of this paper is to study the above problem in a unifying approach using generalized sequence in order to obtain the basic result in general form and then indicate their application to concrete cases for example, **sequence and series of functions, integrals and improper integrals depending on parameters.** The unification will be obtained by the omission of inessential details. Hence, the advantage of such an abstract approach is that it concentrates on the essential facts, so that these facts become clearly visible, since the investigator's attention is not disturbed by unimportant details. It is quite observed fact that the abstract method is the simplest and most economical method for treating mathematical systems.

First, we identify our problems, give the definition of pointwise convergence and uniform convergence, then introduce the Cauchy condition for uniform convergence of generalized sequence of functions and formulate this into different concrete cases (Klippert & Williams, 2002). Similarly, we will study Weierstrass test, repeated and double limits.

Let X and T be any sets in a metric space. Consider a function given by

$$y := F(x,t)$$

with values $y \in R^n$, depending on variables $x \in X$ and $t \in T$. By fixing one of the variables, say $t = t_0$ we obtain a function of the single variable x such that

$$F_{t_0}(x) := F(x,t_0).$$

Then changing t_0, we obtain a family of functions

$$F_t(x) := F(x,t)$$

depending on a parameter t. Analogously, we may obtain a second family of functions of a variable t such that

$$F_x(t) := F(x,t).$$

Assume that T is a partially ordered set directed to the right without maximal elements. By fixing x, we may consider the limit

$$\lim_{(T,\geqslant)} F(x,t) = \lim_{(T,\geqslant)} F_t(x).$$

In short, we write

$$\lim_t F(x,t) = \lim_t F_t(x).$$

We will study the problem raised above on the basis of this limit of a generalized sequence of functions, i.e., family of functions on the set T directed to the right.

Definition (*Pointwise Convergence)*
If for some $x \in X$, $\lim_{t} F(x,t)$ *exists, then x is called a point of convergence of the function* $F(x,t)$ *with respect to* t *.*
The set of all points of convergence is called the domain of convergence. Let $D \subseteq X$ be a domain of convergence. Then the limit under consideration is a function of $x \in X$. Define

$$\varphi(x) = \lim_{t} F(x,t) \ .$$

In this case, we say that $F(x,t)$ converges pointwise to $\varphi(x)$ in X and write

$$F(x,t) \to \varphi(x) \text{ on } D.$$

Thus

$$F(x,t) \to \varphi(x) \Leftrightarrow$$
$$\forall x \in X, \forall \varepsilon > o, \exists t_o \in T : \forall t \geqslant t_0$$
$$| F(x,t) - \varphi(x) | < \varepsilon \ .$$

One of the basic problems being studied in the theory of uniform convergence is to obtain results on properties of the limit function $\varphi(x)$ depending on the properties of family $F_t(x)$, i.e., on the properties of function $F(x,t)$ regarded as a function of *x*.

If $X \subseteq R^m$, then we may study conditions for continuity, integrability and differentiability of $\varphi(x)$ depending on corresponding properties of $F(x,t)$. Moreover, if the set X is assumed to be directed to the right, then the directed ordering on $X \times T$ can be defined in the usual way. In this case, we face with the problem of equality of three limits – two repeated limits and one double limit:

$$\lim_{x} \lim_{t} F(x,t), \lim_{t} \lim_{x} F(x,t), \lim_{x,t} F(x,t)$$

Here are some examples (Bajracharya & Phulara, 2008) of the situation leading to the problems raised above.

1. Sequence of Functions: Let $X = \boldsymbol{R}$ and $T = N$. We define a sequence $\{f_n\}$ of functions on *X*. Assume that

$$\forall x \in X \qquad \lim_{n\to\infty} f_n(x) = \varphi(x).$$

We face the following problems:
If each f_n *continuous on X, is* φ *continuous on X?*
Can we assert

$$\int_a^b \varphi(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx \ ?$$

Can we obtain

$$\varphi'(x) = \lim_{n\to\infty} f'_n(x) ?$$

If it is taken into account that continuity, differentiability and integrability themselves are certain limiting processes, the problems raised above can be viewed as a problem of interchanging two limiting processes in general.

2. Series of functions: Given a series $\sum_{n=1}^{\infty} f_n(x)$ of functions under what conditions is its sum continuous (or differentiable or integrable) if each term of the series is continuous (accordingly differentiable or integrable)? This question is also related to the problem raised above, since the sum of a series is defined as the limit of *n*-th partial sum of the series,

$$S(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) .$$

3. Integrals: Let $f(x,y)$ be a function defined for all $x \in [a,b]$ and $y \in [c,d]$. If it is integrable with respect to $x \in [a,b]$ for each fixed $y \in [c,d]$, then the integral

$$\int_a^b f(x,y)dx = \varphi(y)$$

is a function of $y \in [c,d]$. Here also, we can raise the same question as above. Is the function φ continuous or differentiable or integrable with respect to *y* according as the function $f(x,y)$ is continuous, or differentiable, or integrable regarding it as a function of y for a fixed *x* or as a function of vector (x,y)?

4. Improper Integrals Depending on Parameters:

Let us consider an improper integral $\int_a^{\infty} f(x,y)dx$ depending on a parameter *y*. By definition,

$$\int_a^\infty f(x,y)dx = \lim_{t\to\infty}\int_a^t f(x,y)dx.$$

In this case also, we face the same problem as above. Even though the above problems to be raised in varieties of situations may have specific characteristics, they are studied in terms of an important notion of uniform convergence (Gorden, 2000).

Definition (*Uniform-Convergence*)

A function $F(x,t)$ is said to converge uniformly to $\varphi(x)$ with respect to t on X if

$$\forall\varepsilon > 0, \exists\, \tilde{t} \in T : \forall x \in X, \forall t \geqslant \tilde{t}$$

$$| F(x,t) - \varphi(x) | < \varepsilon ,$$

i.e. $\forall\varepsilon > 0, \exists\, \tilde{t} \in T : \forall t \geqslant \tilde{t}$

$\sup_X F(x,t) - \varphi(x) | < \varepsilon$

... (1)

In this writing, the modulus sign signifies the absolute value of a real number or the length of a vector or, the norm of a vector. We write in this case $F(x,t) \rightrightarrows \varphi(x)$ for $x \in X$.

Note that uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform-convergence (Apostol, 2002).

Example : 1

Let $X = [0,1]$ *and* T=N. Then a function

$$F(x,t) = \frac{tx}{1+t^2x^2}$$

converges pointwise but not uniformly on *X*.

Solution

Let $\varphi(x)$ denote the pointwise limit of $F(x,t)$ then we have

$$\varphi(x) = \lim_t F(x,t) = \lim_t \frac{tx}{1+t^2x^2} = \lim_t \frac{\frac{x}{t}}{x^2 + \frac{1}{t^2}} = 0$$

for all $x \in R$.

Choose $\varepsilon = \frac{1}{3}$ and let t' be an integer and greater than *t* such that $\frac{1}{t'} \in X$.

If $x = \frac{1}{t'}$, $t = t'$; then $tx = 1$.

So,

$$|F(x,t) - \varphi(x)| = \left|\frac{tx}{1+t^2x^2} - 0\right| = \left|\frac{1}{1+1} - 0\right| = \frac{1}{2} > \frac{1}{3} = \varepsilon$$

This contradiction shows that

$$F(x,t) = \frac{tx}{1+t^2x^2}$$

converges pointwise but not uniformly on *X*.

The Cauchy Condition for Uniform Convergence

We note that the definitions of both pointwise convergence and uniform convergence require the limit function to be known. They give us a way of verifying whether a function may be the pointwise limit or the uniform limit of the sequence. But in practice, it may be necessary to know whether the sequence converges or not without knowing the limit beforehand. In analogy with pointwise convergence, one has Cauchy's criterion for uniform convergence of generalized sequence of functions. We can now formulate the Cauchy condition of uniform convergence as follows.

Theorem : 1

The function $F(x,t)$ *converges uniformly to a certain function* $\varphi(x)$ *with respect to t on X if and only if,*

$$\forall\varepsilon > 0,\ \exists\, \tilde{t} \in T : \forall t', t'' \geqslant \tilde{t}, \forall x \in X ,$$

$$|F(x,t') - F(x,t'')| < \varepsilon .$$

Proof

"$\Rightarrow$"

Suppose that $F(x,t) \rightrightarrows \varphi(x)$ on *X*. Then

$$\forall\varepsilon > 0, \exists\, t_0 : \forall t' \geqslant t_0\ \forall x \in X \Rightarrow$$

$$|F(x,t') - \varphi(x)| < \frac{\varepsilon}{2}.$$

Also,

$$\forall t'' \geqslant t_0\ \forall x \in X$$

$$|F(x,t'') - \varphi(x)| < \frac{\varepsilon}{2}.$$

Now by triangle inequality,

$$|F(x,t') - F(x,t'')| \le |F(x,t') - \varphi(x)| + |F(x,t'') - \varphi(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence,

$$\forall x \in \mathrm{X} \ \ |F(x,t') - F(x,t'')| < \varepsilon .$$

$$'' \Leftarrow ''$$

Let $F(x,t)$ be any function on *X* such that

$$\forall \varepsilon > 0 \ \exists \tilde{t} \in T :$$

$$\forall t', t'' \succcurlyeq \tilde{t} \ \ |F(x,t') - F(x,t'')| < \varepsilon \qquad \ldots (2)$$

holds.

We have to show that there is a function φ on *X* such that $F(x,t) \rightrightarrows \varphi$ on *X*.

From *(2)* we see that, for each fixed $x \in \mathrm{X}$, the sequence of real numbers $F(x,t)$ is a Cauchy sequence. Hence $\lim\limits_{t} F(x,t)$ exists for every $x \in \mathrm{X}$ (Note that every Cauchy sequence of real numbers is convergent). Define φ by

$$\varphi(x) = \lim_{t} F(x,t) \quad (x \in \mathrm{X})$$

Keeping t'' fixed in *(2)* and letting $t' \to \infty$, we get

$$\forall \ t'' \succcurlyeq \tilde{t}, \forall \ x \in \mathrm{X}$$

$$| F(x,t') - \varphi(x) | < \varepsilon .$$

Since ε was arbitrary, this shows that

$$F(x,t) \rightrightarrows \varphi(x) \text{ on } X.$$

From the above generalized theorem, now we formulate Cauchy conditions for our four problems. Moreover, the Cauchy condition for uniform convergence of a sequence of functions (Apostol, 2002) as follows.

Theorem : 2

The sequence of functions $\{f_n\}$ *converges uniformly to a certain function* $\varphi(x)$ *on X if and only if,*

$$\forall \varepsilon > 0, \exists N > 0 : \forall m, n > N, \forall x \in X ,$$

$$| f_m(x) - f_n(x) | < \varepsilon .$$

Next, we formulate the Cauchy condition for uniform convergence of a series of functions (Rudin, 1976) as below.

Theorem : 3

The series of functions $\sum f_n(x)$ *converges uniformly to a certain function* $\varphi(x)$ *on X if and only if,*

$$\forall \varepsilon > 0, \exists N > 0 : \forall n > N, \forall x \in X,$$

$$\left| \sum_{k=n=1}^{n+p} f_k(x) - \varphi(x) \right| < \varepsilon$$

for each $p = 1, 2, \ldots$

We now formulate the Cauchy condition for integrals depending on parameters (Bartle & Sherbert, 1994) as below.

Theorem : 4

Let $f(x,y)$ *be a function defined for all* $x \in [a,b]$ … (2)
and $y \in [c,d]$. *If it is integrable with respect to* $x \in [a,b]$ *for each fixed* $y \in [c,d]$, *then there exist a function* $\varphi(y)$ *such that*

$$\int_a^b f(x,y) \rightrightarrows \varphi(y) \ \textit{on X} \Leftrightarrow$$

$$\forall \varepsilon > 0, \exists \tilde{y} \in [c,d] : \forall y', y'' \geq \tilde{y}, \forall x \in [a,b],$$

$$| \int_a^b f(x,y') - f(x,y'') | < \varepsilon .$$

Finally, we formulate the Cauchy condition for improper integrals depending on parameters (Gorden, 2000) as below.

Theorem : 5

Let $\int_a^\infty f(x,y)dx$ *be an improper integral depending on a parameter y. Then there exist a function* $\lim\limits_{t} \int_a^t f(x,y)dx$ *such that*

$$\int_a^\infty f(x,y) \rightrightarrows \lim_{t} \int_a^t f(x,y)dx \text{ on } X$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists y : \forall y', y'' > y, \forall x \in X ,$$

$$| \int_a^\infty f(x,y') - f(x,y'') | < \varepsilon .$$

Weierstrass Test

In general, it is neither easy to prove that a given sequence of functions is convergent by Cauchy's criterion nor even from the definitions. In such cases the following theorem called Weierstrass test is a useful tool for uniform convergence of generalized sequence of functions. The Weierstrass test for uniform convergence provides us only the sufficient conditions for uniform convergence.

Theorem : 6

Let φ *be a function on T such that* $\lim\limits_{t} \varphi(t)$ *exists and*

$$\forall t', t'' \in T, \ \forall x \in X ,$$

$$|F(x,t') - F(x,t'')| \leq |\varphi(t'') - \varphi(t')|.$$

Then $F(x,t)$ *converges uniformly on* X.

Proof

Since $\varphi(t)$ is convergent sequence, for a given $\varepsilon > 0$, we can find a set directed to the right T such that

$$\forall t', t'' \in T, \ |\varphi(t') - \varphi(t'')| < \varepsilon.$$

Now,

$$\forall x \in X, \forall t', t'' \in T,$$
$$|F(x,t') - F(x,t'')| \leq |\varphi(t') - \varphi(t'')| < \varepsilon.$$

Hence $F(x,t)$ converges uniformly on X.

From the above generalized theorem, now we formulate the Weierstrass test for our four problems. Moreover, first we formulate it to a sequence of functions (Klippert & Williams, 2002) as follows.

Theorem : 7

Let φ *be a function on X such that* $\lim_n \varphi(x)$ *exists and*

$$\forall m, n \in N, \ \forall \ x \in X,$$
$$|f_n(x) - f_m(x)| \leq \varphi(x).$$

Then $f_n(x)$ *converges uniformly on X.*

Next, we formulate it into a series of functions (Osgood, 1897) as below.

Theorem : 8

Let φ_n *be a sequence of positive real numbers such that*

$$|f_n(x)| \leq \varphi_n \ for \ x \in X, n \in N.$$

If the series $\sum \varphi_n$ *is convergent, then* $\sum f_n(x)$ *is uniformly convergent on X.*

Now, we formulate the Weierstrass test for integrals depending on parameters (Luxemberg, 1971) as below.

Theorem : 9

Let $\varphi(y)$ *be a function on Y such that* $\lim_y \varphi(y)$ *exists and*

$$\forall y', y'' \in Y, \forall x \in X,$$
$$\left|\int_a^b f(x,y') - f(x,y'')\right| \leq \varphi(y).$$

Then $\int_a^b f(x,y)$ *converges uniformly on X.*

Finally, we formulate the Weierstrass test for improper integrals depending on parameters (Cunningham, 1967) as below.

Theorem :10

Let $\varphi(y)$ *be a function on Y such that* $\lim_y \varphi(y)$ *exists and*

$$\left|\int_a^\infty [f(x,y) - \lim_t \int_a^t f(x,y)]dx\right| \leq \varphi(y).$$

Then $\int_a^\infty f(x,y)$ *converges uniformly on X.*

Repeated and Double Limits

Suppose X is a set also directed to the right without maximal elements. The partial orderings in T and X may, in general, be different. In the Cartesian product, the strict partial ordering is defined in the standard way (Egan, 1940) as follows:

$$(x_1,t_1) \succ (x_2,t_2) \Leftrightarrow x_1 \succ x_2 \text{ and } t_1 \succ t_2$$

.

Now the limits,

$$\lim_x \lim_t F(x,t), \ \lim_t \lim_x F(x,t), \ \lim_{(x,t)} F(x,t)$$

will be meaningful.

It is well-known that the existence of the double limit does not imply the existence of the repeated limits, and the converse does not hold, as well. Moreover, the repeated limits may not be equal, even if they both exist. Consider the following theorem.

Theorem : 11

If $\lim_{(x,t)} F(x,t) = L$ *and* $\lim_t F(x,t) = \phi$, *then* $\lim_x \lim_t F(x,t)$ *exists and*

$$\lim_x \lim_t F(x,t) = \lim_x \varphi(x) = L.$$

Proof

Let $\lim_{(x,t)} F(x,t) = L$. Given $\varepsilon > 0$, choose t_0 such that

$$\forall \ t \succ t_0, \ |F(x,t) - L| < \frac{\varepsilon}{2}. \qquad \text{... (3)}$$

Let $\varphi(x) = \lim_t F(x,t)$, so for each fixed x, the limit $\lim_{x} F(x,t)$ exists. Then choose t_1 such that

$$\forall\, t \succ t_1,\ |\varphi(x) - F(x,t)| < \frac{\varepsilon}{2}\,.$$

Since t_1 depends upon x as well as on ε. For each $x \succ t_0$, we choose t_1 and then choose a fixed t greater than t_0 and t_1. Then by the triangle inequality, we get

$$|\varphi(x) - L| \le |\varphi(x) - F(x,t)| + |F(x,t) - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence,

$$\forall\, x \succ t_0,\ |\varphi(x) - L| < \varepsilon\,.$$

$$\therefore \lim_{x} \varphi(x) = L\,.$$

i.e. $\lim_{x} \lim_{t} F(x,t) = \lim_{x} \varphi(x) = L$.

Thus, $\lim_{x} \lim_{t} F(x,t)$ exists and

$$\lim_{x} \lim_{t} F(x,t) = \lim_{x} \varphi(x) = L\,.$$

Note that from the above theorem, the existence of the double limit $\lim_{(x,t)} F(x,t)$ implies the existence of the iterated limit $\lim_{x}\left(\lim_{t} F(x,t)\right)$. The following example (Bajracharya & Phulara, 2008) shows that the converse of the above theorem may not be true.

Example : 2

Let us consider a double sequence

$$F(x,t) = \frac{xt}{x^2 + t^2}\,.$$

Solution

We have

$$F(x,t) = \frac{xt}{x^2 + t^2}$$

Now,

$$\lim_{t} F(x,t) = \lim_{t} \frac{xt}{x^2 + t^2}$$

$$= \lim_{t} \frac{xt}{t\left[\frac{x^2}{t} + t\right]} = 0$$

$$\therefore \lim_{x}\left(\lim_{t} F(x,t)\right) = 0\,.$$

Similarly,

We can find $\lim_{t}\left(\lim_{x} F(x,t)\right) = 0$.

But when $x = t$, ... (4)

$$F(x,t) = \frac{xt}{x^2 + t^2} = \frac{t^2}{2t^2} = \frac{1}{2}\,.$$

And when $x = 2t$,

$$F(x,t) = \frac{xt}{x^2 + t^2} = \frac{2t^2}{5t^2} = \frac{2}{5}\,.$$

Hence in different directions, the double limits are different. So, the double limit cannot exist in this case. We now present two theorems in which uniform convergence is involved.

Theorem : 12

Let $F(x,t)$ *converges uniformly with respect to t to* $\varphi(x)$ *on* X. *If* $\lim_{x} \varphi(x) = A$, *then*

$$\lim_{(x,t)} F(x,t) = L\,.$$

Proof

Let ε be a positive number. Then there exist t_0 such that if $t \succ t_0$, then

$$\forall\, x \in A,\ \rho(F(x,t), \varphi(x)) < \frac{\varepsilon}{2}\,.$$

Then there exist x_0 *in* X such that

$$\forall\, x \succ x_0,\ \rho(\varphi(x), L) < \frac{\varepsilon}{2}\,.$$

Now,

If $x \succ x_0$ and $t \succ t_0$, both these hold. So

$$\rho(F(x,t), L) \le \rho(F(x,t), \varphi(x)) + \rho(\varphi(x), L)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

In other words, if the repeated limit exists and the inner limit fulfils the requirements of uniform convergence, then the double limit exists, and equals to the repeated limit.

Theorem : 13

Let $F(x,t)$ *converge uniformly with respect to* t *to* $\varphi(x)$ *and* $\lim_{x} F(x,t) = \psi(t)$. *Then all three limits exist and*

$$\lim_{x} \lim_{t} F(x,t) = \lim_{t} \lim_{x} F(x,t) = \lim_{(x,t)} F(x,t)\,.$$

Proof

Let $\varepsilon > 0$. There exist t_0 in T such that if $t \succ t_0, x \; \varepsilon \; X$,

$$\rho(F(x,t), \varphi(x)) < \frac{\varepsilon}{6}.$$

Let t^* be a point in T. Then there exists x_0 in X such that if $x \succ x_0$, then

$$\rho(F(x,t^*), \psi(t^*)) < \frac{\varepsilon}{6}.$$

Now,

Let (x', t') and (x'', t'') be any two points in $X \times T$.

Then $\rho(F(x',t'), F(x'',t'') \le \rho(F(x',t')), \varphi(x')$

$+ \rho(\varphi(x'),$

$F(x',t^*)) + \rho(F(x',t^*), \psi(t^*))$

$+ \rho(\psi(t^*), F(x'',t^*)) + \rho(F(x'',t^*), \psi(x''))$

$+ \rho(\psi(x''), F(x'',t''))$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon.$$

By Theorem 11,

$$\lim_x \lim_t F(x,t) = \lim_t \lim_x F(x,t) = \lim_{(x,t)} F(x,t)$$

CONCLUSIONS

Generalized sequence of functions in order to obtain the basic result in general form and then indicated their application to concrete cases. Also, we have analyzed that pointwise convergence is usually not strong enough to transfer any of the properties mentioned from the individual term of sequence to the limit function. Therefore, we studied stronger methods of convergence that preserve certain properties such as continuity, differentiability and integrability. Among them, the notion of uniform convergence plays a central role. The method which we use is the simplest, most economical and beneficial for those who are interested in real analysis. If the sequence of functions is neither convergent by *Cauchy*'s criterion nor by definition, then in such a situation, *Weierstrass* test plays a significant role in uniform convergence of generalized sequence of functions. The uniform convergence of the generalized sequence of continuous functions is sufficient to guarantee the continuity of its limit function, but it is not necessary. Similar results arrived in the case of differentiation and integration respectively.

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CONFLICT OF INTERESTS

The authors declare no conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the author, upon reasonable request.

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