Exponentiated–Exponential Logistic Distribution: Some Properties and Application

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Abstract

This study proposes new distribution which is generated from exponentiated-exponential-X family of distribution. It is explored various shape and behavior of the observed distribution through probability density plot, hazard rate function and quantile function. Further we have investigated some mathematical properties, estimation of the parameters and associated confidence interval using maximum likelihood estimation (MLE) method of the exponentiated-exponential-logistic distribution (EELD).

Keywords: Exponentiated-exponential-X family, Hazard function, Logistic distribution, MLE.

Introduction

In the last few years, new generated families of continuous distributions have attracted several statisticians to develop new models. To describe the real world phenomena, we generally use statistical distribution. Since real data are usually complex and they have a variety of shapes, existing distribution do not always provide an adequate fit. Hence investigating new distribution and studying their behavior and flexibility are of interest of researchers for last decades. To improve the flexibility of the statistical model, the families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution.

In probability and statistical modeling these families have been broadly studied in several areas as well as produced more flexibility in many applications. Some of the generating family of the distributions are: Beta-generated (Eugene et al., 2002), Gamma-generated-G family was defined by (Zografos & Balakrishnan, 2009), the gamma-exponentiated exponential distribution by (Ristic & Balakrishnan, 2011), a new family of generalized distribution was introduced by (Cordeiro & de Castro, 2011), exponentiated generalized class introduced by (Cordeiro et al., 2013), Weibull-G by (Bourguignon et al., 2014), Similarly, Kumaraswamy Weibull-G was defined by (Hassan and Elgarhy, 2016), exponentiated Weibull-G family by (Hassan & Elgarhy, 2016), additive Weibull- G family by (Hassan & Hemeda, 2016), exponentiated extended-G (Elgarhy et al., 2017), generalized additive Weibull-G (Hassan et al., 2017), power Lindley-G (Hassan & Nassr (2018) and Muth-G (Almarashi & Elgarhy, 2018). Elgarhy et al. (2019) have defined Type II half logistic exponential distribution. Abdulkabir and Ipinyomi (2020) have introduced a three parameter Type II half logistic exponentiated exponential distribution.

Alzaatreh et al. (2013) has introduced a beta-exponential-X family whose probability density function (PDF) and cumulative density function (CDF) are,

$$g(x)=\frac{\lambda}{\mathrm{B}(\alpha,\beta)}f(x)\left[1-F(x)\right]^{\lambda\beta-1}\left[1-\left[1-F(x)\right]^{\lambda}\right]^{\alpha-1} \tag{1.1}$$

and $G(x)=1-I_{\left[1-F(x)\right]^{\lambda}}\left[\lambda(\beta-1)+1,\alpha\right]$ (1.2)

respectively, where I is incomplete beta function. When $b=1$ in (1.1) and (1.2) it reduces to exponentiated-exponential-X family with PDF and CDF,

$$g(x)=\alpha\lambda f(x)\left[1-F(x)\right]^{\lambda-1}\left[1-\left[1-F(x)\right]^{\lambda}\right]^{\alpha-1} \tag{1.3}$$

and $G(x)=\left[1-\left[1-F(x)\right]^{\lambda}\right]^{\alpha}$ (1.4)

respectively.

The Exponentiated-Exponential-Logistic Distribution (EELD)

In probability theory and statistics, the logistic distribution is a important continuous probability distribution. Its cumulative distribution function is the logistic function, which appears in logistic regression. It resembles the normal distribution in shape but has heavier tails (higher kurtosis). The logistic distribution is a special case of the Tukey lambda distribution. The probability density function (PDF) and cumulative density function (CDF) of standard logistic distribution are

$$f\left(x\right)=\frac{e^{-x}}{\left(1+e^{-x}\right)^{2}};\quad -\infty<x<\infty \tag{2.1}$$

and
$$F\left(x\right)=\frac{1}{\left(1+e^{-x}\right)};\quad -\infty<x<\infty \tag{2.2}$$

respectively. After plug in PDF and CDF of standard logistic distribution in (1.3) and (1.4) we get PDF and CDF of new distribution **exponentiated-exponential-logistic (EEL) distribution** and they can be defined as

$$g\left(x\right)=\frac{\lambda\alpha e^{-\lambda x}}{\left(1+e^{-x}\right)^{\lambda+1}}\left[1-\frac{e^{-\lambda x}}{\left(1+e^{-x}\right)^{\lambda}}\right]^{\alpha-1};\quad -\infty<x<\infty;\alpha,\lambda>0 \tag{2.3}$$

and
$$G\left(x\right)=\left[1-\left[1+e^{x}\right]^{-\lambda}\right]^{\alpha};\quad -\infty<x<\infty;\alpha,\lambda>0 \tag{2.4}$$

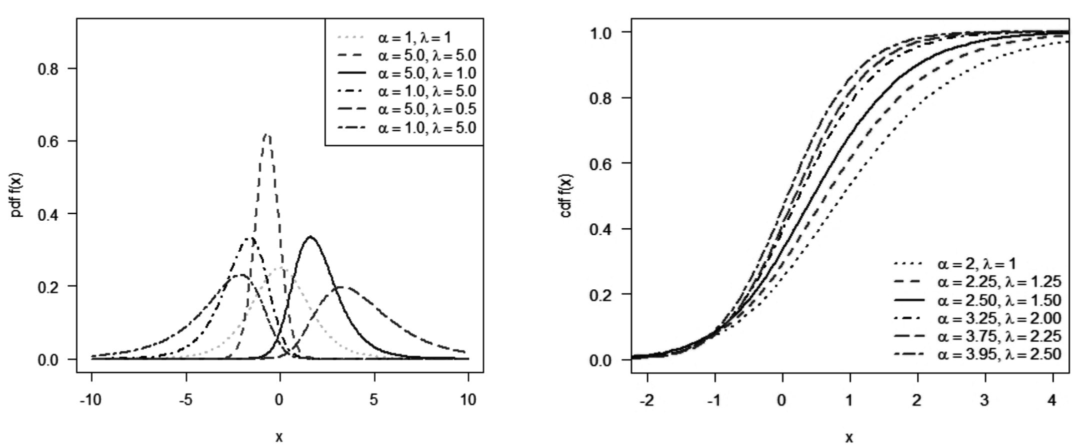
Special cases of EELD

I. When λ = 1, the EELD in (2.3) reduces to type-I generalized logistic distribution given by (Johnson et al. 1995).

II. When α = λ = 1, in (2.3) it reduces to standard logistic distribution.

Figure 1

*Plots of the Probability Density Function (left panel) and Cumulative Density Function (right panel) for Different Values of **α** and **λ***



Reliability/ Survival function

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive beyond any specified time. The survival function is also known as the survivor function or reliability function. Let T be a continuous random variable with cumulative distribution function $F(t)$ on the interval $[0, \infty)$. Its survival function or reliability function is:

$$S(t) = p(T > t) = \int_t^\infty f(u)du = 1 - F(t)$$

$$= 1 - \left[1 - \left[1 + e^t\right]^{-\lambda}\right]^\alpha ;\ 0 < t < \infty; \alpha, \lambda > 0 \qquad (3.1)$$

Hazard Function

Suppose that an item has survived for a time t and we desire the probability that it will not survive for an additional time *dt* then, hazard rate function is,

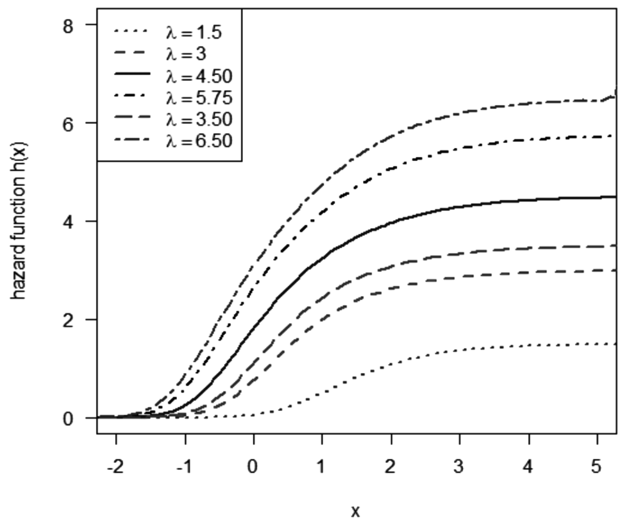
$$h(t) = \lim_{dt \to 0} \frac{pr(t \le T < t + dt)}{dt.S(t)} = \frac{f(t)}{S(t)} = \frac{f(t)}{1 - F(t)};\ 0 < t < \infty \qquad (4.1)$$

Now the hazard function for EEL distribution is

$$h(x) = \frac{g(x)}{1 - G(x)} = \frac{\dfrac{\lambda \alpha e^{-\lambda x}}{\left(1 + e^{-x}\right)^{\lambda + 1}} \left[1 - \dfrac{e^{-\lambda x}}{\left(1 + e^{-x}\right)^{\lambda}}\right]^{\alpha - 1}}{1 - \left[1 - \left[1 + e^x\right]^{-\lambda}\right]^\alpha};\ 0 < x < \infty; \alpha, \lambda > 0 \qquad (4.2)$$

Figure 2

Plots of the Hazard Function of EELD for Different Values of Shape Parameter λ Keeping α Constant



The cumulative Hazard Function

The cumulative hazard function (CHF) of EELD is

$$\begin{aligned} H(x) &= -\log\left[1-F(x)\right] \\ &= -\log\left\{1-\left[1-\left[1+e^{x}\right]^{-\lambda}\right]^{\alpha}\right\};\ 0<x<\infty;\alpha,\lambda>0 \end{aligned} \tag{4.1.1}$$

Quantile Function of PC Distribution

In probability and statistics, the quantile function, associated with a probability distribution of a random variable, specifies the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. It is also called the percent-point function or inverse cumulative distribution function.

$Q(p) = F^{-1}(P)$

The quantile function is

$$Q(p) = \ln\left[\left(1+p^{1/\alpha}\right)^{-1/\lambda} - 1\right]; 0<p<1 \tag{5.1}$$

For the generation of the random numbers of the EEL distribution, we suppose simulating values of random variable X with the CDF (2.4). Let V denote a uniform random variable in (0,1), then the simulated values of X are obtained by setting,

$$x = \ln\left[\left(1+v^{1/\alpha}\right)^{-1/\lambda} - 1\right]; 0 < v < 1 \tag{5.2}$$

Skewness and Kurtosis

The Skewness and Kurtosis are used mostly in data analysis to study the shape of the probability distribution or data set. Which can be calculated as follows,

$$S_k(B) = \frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)}\text{ , and}$$

The coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_u(Moors) = \frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

Maximum Likelihood Estimation (MLE)

In this section we illustrated the maximum likelihood estimators (MLE's) of the EEL distribution.

Let $\underline{x}$ = $(x_1, \ldots\ldots x_n)$ be a random sample of size 'n' from *EEL* (α,λ) the log-likelihood function $L(\alpha,\lambda \mid \underline{x})$ an be written as,

$$L(\alpha,\lambda/\underline{x}) = n\ln(\alpha\lambda) - \lambda\sum_{i=1}^{n} x_i - (\lambda+1)\sum_{i=1}^{n}\ln(1+e^{-x_i}) + (\alpha-1)\sum_{i=1}^{n}\ln\left\{1 - \frac{e^{-\lambda x_i}}{\left(1+e^{-x_i}\right)^{\lambda}}\right\} \tag{7.1}$$

The maximum likelihood estimators of the parameters have obtained by differentiating (7.1) with respect to parameters α and λ and equating to zero, we have

$$\frac{\delta L}{\delta\alpha} = 1 + \frac{1}{\alpha} - \frac{e^{-\lambda x}}{(1+e^{-x})^{\lambda}} = 0 \tag{7.2}$$

$$\frac{\delta L}{\delta\alpha} = 1 + \frac{1}{\alpha} - \frac{e^{-\lambda x}}{(1+e^{-x})^{\lambda}} = 0 \tag{7.3}$$

After solving these two nonlinear equations (7.2) and (7.3) we will get the maximum likelihood estimator $\hat{\omega} = \left(\hat{\alpha}, \hat{\lambda}\right)^T$ *of* $\omega = \left(\alpha, \lambda\right)^T$. But these equations cannot be solved analytically and statistical software can be used to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to calculate the estimate $\hat{\omega}$. For example, *optim()* function in R software can be used to compute $\hat{\omega}$ numerically.

Hence, from the asymptotic normality of MLEs, approximate (1- α)% confidence intervals (ACI) for α and λ can be constructed as

$$\hat{\alpha} \pm Z_{\alpha/2} SE\left(\hat{\alpha}\right) \text{ and } \hat{\lambda} \pm Z_{\alpha/2} SE\left(\hat{\lambda}\right) \tag{7.4}$$

where $Z_{a/2}$ is the upper percentile of standard normal variate.

Illustration with Real Dataset

Here we present the estimated values of model parameters and approximate confidence interval (ACI) by using a real data set. The real data (sorted) set represents the remission times (in months) of a random sample of 128 bladder cancer patients used by (Lee & Wang, 2003) 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28,7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

The ML estimates are computed by maximizing the log-likelihood function given in equation (7.1) directly using optim() function in R software (R Development Core Team, 2020) and (Mailund, 2017). We obtain $\hat{a}$ = 1.4159 and $\hat{l}$ = 0.13232 and the corresponding log-likelihood value is l = -418.4075.

Table 1

MLE, SE and 95% Confidence Interval

Parameter	MLE	SE	95% ACI	t-value	p-value
alpha	1.4159	0.18031	(1.0625, 1.7693)	7.853	4.07e-15
lambda	0.13232	0.01448	(0.1040, 0.1607)	9.141	< 2e-16

Conclusion

In this study, we introduced a two parameter new distribution called Exponentiated Exponential Logistic (EEL) Distribution generated by a new class of Exponentiated Exponential-X family of distribution. We have derived important properties of the EEL distribution like reliability function, hazard rate function, quantile function and maximum likelihood estimation of parameters and their associated confidence intervals. We have explored the application of EEL distribution to a real data set used by earlier researchers. The purposed model demonstrated the flexibility in its shape so it may be an alternative model in the fields of reliability/survival analysis.

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