COMMON COUPLED FIXED POINTS FOR TWO PAIRS OF \textit{w}-COMPATIBLE MAPS IN PARTIAL $G$-METRIC SPACES

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ABSTRACT
In this paper we prove a unique common coupled fixed point theorem for two pairs of \textit{w}-compatible mappings satisfying two contractive conditions in partial $G$-metric spaces. We also furnish an example to support our main theorem.

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INTRODUCTION
Dhage [5] introduced the concept of $D$-metric spaces to generalize the ordinary metric spaces and proved several results, for example, refer [5, 6, 7]. Unfortunately almost all results are invalid (see [19, 20, 21, 13, 15]). To modify $D$-metric space, Mustafa and Sims [13] introduced the concept of $G$-metric spaces and obtained some results in their papers. Later several authors, for instance, [4, 10, 2, 3, 22, 24, 25, 26, 9, 14, 16, 17, 18], proved some fixed, common fixed and coupled fixed point theorems in $G$-metric spaces.

Recently Salimi and Vetro [23] defined partial $G$-metric spaces using the concept of partial metric spaces introduced by Mathews [12].

Kaewcharoen [10] proved a unique common fixed point theorem for four self mappings on a $G$-complete metric spaces. The intent of this paper is to extend the theorem of kaewcharoen [10] in partial $G$-metric spaces. We illustrated our theorem with an example.

First we state the following known definitions, lemmas and propositions.

**Definition 1.1 [5]:** Let $X$ be a non-empty set. A $D$-metric on $X$ is a function $D: X^3 \to [0, +\infty)$
that satisfies the following conditions for each \( x, y, z, a \in X \),

1. \( D(x, y, z) = 0 \) if and only if \( x = y = z \),
2. \( D(x, y, z) = D(p(x, y, z)) \) where \( p \) is a permutation function,
3. \( D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z) \).

Then the pair \((X, D)\) is called a \( D \)-metric space.

**Definition 1.2 [13]**: Let \( X \) be a non-empty set and let \( G : X \times X \times X \to [0, \infty) \) be a function satisfying the following properties:

\[
\begin{align*}
(G_1) & : \quad G(x, y, z) = 0 \quad \text{if} \quad x = y = z, \\
(G_2) & : \quad 0 < G(x, x, y) \quad \text{for all} \quad x, y \in X \quad \text{with} \quad x \neq y, \\
(G_3) & : \quad G(x, x, y) \leq G(x, y, z) \quad \text{for all} \quad x, y, z \in X \quad \text{with} \quad y \neq z, \\
(G_4) & : \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots, \quad \text{symmetry in all three variables}, \\
(G_5) & : \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad \text{for all} \quad x, y, z, a \in X.
\end{align*}
\]

Then the function \( G \) is called a generalized metric or a \( G \)-metric on \( X \) and the pair \((X, G)\) is called a \( G \)-metric space.

**Definition 1.3 [12]**: A partial metric on a non-empty set \( X \) is a function \( p : X \times X \to [0, \infty) \) such that for all \( x, y, z \in X \),

\[
\begin{align*}
(p_1) & : \quad x = y \iff p(x, x) = p(x, y) = p(y, y), \\
(p_2) & : \quad p(x, x) \leq p(x, y), p(y, y) \leq p(x, y), \\
(p_3) & : \quad p(x, y) = p(y, x), \\
(p_4) & : \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).
\end{align*}
\]

The pair \((X, p)\) is called a partial metric space (PMS).

**Definition 1.4 [23]**: Let \( X \) be a non-empty set and let \( P : X \times X \times X \to [0, +\infty) \) be called a partial \( G \)-metric if the following conditions are satisfied:
(P1) If \( x = y = z \) then \( P(x, y, z) = P(x, x, x) = P(y, y, y) = P(z, z, z) \),
(P2) \( P(x, x, x) + P(y, y, y) + P(z, z, z) \leq 3P(x, y, z) \) for all \( x, y, z \in X \),
(P3) \( \frac{1}{3}P(x, x, x) + \frac{2}{3}P(y, y, y) < P(x, y, y) \) for all \( x, y \in X \) with \( x \neq y \),
(P4) \( P(x, x, y) = \frac{1}{3}P(x, x, x) \leq P(x, y, z) = \frac{1}{3}P(z, z, z) \) for all points \( x, y, z \in X \) with \( y \neq z \),
(P5) \( P(x, y, z) = P(x, z, y) = P(y, z, x) = \cdots \) (symmetry in three variables),
(P6) \( P(x, y, z) \leq P(x, a, a) + P(y, a, a) - P(a, a, a) \) for any \( x, y, z, a \in X \).

Then the pair \((X, P)\) is called a partial \( G \)-metric space (in brief PGMS).

**Example 1.1 [23]:** Let \( X = [0, +\infty) \) and define \( P(x, y, z) = \frac{1}{3}(\max\{x, y\} + \max\{y, z\} + \max\{x, z\}) \) for all points \( x, y, z \in X \). Then \((X, P)\) is a PGMS.

The following Proposition gives some properties of a partial \( G \)-metric.

**Proposition 1.1 [23]:** Let \((X, P)\) be a PGMS. Then for \( x, y, z, a \in X \), the following properties hold:

1. If \( P(x, y, z) = P(x, x, x) = P(y, y, y) = P(z, z, z) \), then \( x = y = z \)
2. If \( P(x, y, z) = 0 \) then \( x = y = z \);
3. If \( x \neq y \), then \( P(x, y, y) > 0 \)
4. \( P(x, y, z) \leq P(x, x, x) + P(x, x, z) - P(x, x, x) \) for any \( x, y, z, a \in X \).
5. \( P(x, y, y) \leq 2P(x, x, y) - P(x, x, x) \);
6. \( P(x, y, z) \leq P(x, a, a) + P(y, a, a) - P(a, a, a) - 2P(a, a, a) \);
7. \( P(x, y, z) \leq P(x, a, z) + P(a, y, z) - \frac{2}{3}P(a, a, a) - \frac{1}{3}P(z, z, z) \) with \( y \neq z \);
Definition 1.5 [23]: Let \((X, P)\) be a PGMS. Then

1. A sequence \(\{x_n\}\) is \(P-G\)-convergent to \(x \in X\) if and only if

\[
P(x, x, x) = \lim_{n \to \infty} P(x, x, x_n) = \lim_{n \to \infty} P(x, x_n, x).
\]

2. A sequence \(\{x_n\}\) is \(0-P-G\)-Cauchy if and only if

\[
\lim_{m, n \to \infty} P(x_n, x_m, x_n) = 0.
\]

3. A partial \(G\)-metric space \((X, P)\) is said to be \(0-P-G\)-complete if and only if every \(0-P-G\)-Cauchy sequence in \(X\) \(P-G\)-converges to a point \(x \in X\) such that \(P(x, x, x) = 0\).

Example 1.2 [23]: Let \(X = [0,1]\) and \(P: X^3 \to [0, \infty)\) be defined by

\[
P(x, y, z) = \max\{x, y\} + \max\{y, z\} + \max\{x, z\}\]

for all points \(x, y, z \in X\). Then \((X, P)\) is a \(0-P-G\)-complete partial \(G\)-metric space.

Lemma 1.1 [23]: Let \((X, P)\) be a partial \(G\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Assume that \(\{x_n\}\) \(P-G\)-converges to \(x \in X\) and \(P(x, x, x) = 0\). Then \(\lim_{n \to \infty} P(x_n, y, y) = P(x, y, y)\) for all \(y \in X\).

Similarly, we can have the following Lemma.

Lemma 1.2: Let \((X, P)\) be a partial \(G\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Assume that \(\{x_n\}\) \(P-G\)-converges to \(x \in X\) and \(P(x, x, x) = 0\). Then \(\lim_{n \to \infty} P(x_n, x_n, y) = P(x, x, y)\) for all \(y \in X\).

Bhskar and Lakshmikantham [8] developed some coupled fixed point theorems for a mapping satisfying mixed monotone property in partially ordered metric spaces. Later Lakshmikantham and Ciric [11] extended the notion of mixed monotone property to mixed g-monotone property and generalized the results of [8]. Abbas et al. [1] introduced \(w\)-compatible mappings and proved some common coupled fixed point theorems in cone metric spaces.

Definition 1.6 [8]: An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F: X \times X \to X\) if \(x = F(x, y)\) and \(y = F(y, x)\).

Definition 1.7 [11]: An element \((x, y) \in X \times X\) is called
(i) a coupled coincident point of mappings \( F : X \times X \to X \) and \( f : X \to X \) if \( fx = F(x, y) \) and \( fy = F(y, x) \).

(ii) a common coupled fixed point of mappings \( F : X \times X \to X \) and \( f : X \to X \) if \( x = fx = F(x, y) \) and \( y = fy = F(y, x) \).

**Definition 1.8 [1]:** The mappings \( F : X \times X \to X \) and \( f : X \to X \) are called a \( w \)-compatible pair if \( f(F(x, y)) = F(fx, fy) \) and \( f(F(y, x)) = F(fy, fx) \) whenever \( fx = F(x, y) \) and \( fy = F(y, x) \).

In 2012, A. Kaewcharoen [10] proved the following

**Theorem 1.1 (Theorem 2.1, [10]):** Let \( X \) be a \( G \)-complete metric space. Suppose that \( \{f, S\} \) and \( \{g, T\} \) are weakly compatible pairs of self-mappings on \( X \) satisfying

\[
G(fx, fx, gy) \leq h \max \left\{ G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), \frac{1}{2} (G(fx, Ty) + G(gy, Sx)) \right\}
\]

and

\[
G(fx, gy, gy) \leq h \max \left\{ G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), \frac{1}{2} (G(fx, Ty) + G(gy, Sx, Sx)) \right\}
\]

for all \( x, y \in X \) where \( h \in \left[ 0, \frac{1}{2} \right) \). Suppose that \( fX \subseteq TX \) and \( gX \subseteq SX \). If one of \( TX \) or \( SX \) is a \( G \)-closed subspace of \( X \), then \( f, g, S \) and \( T \) have a unique common fixed point.

Now we give our main result.

**MAIN RESULT**

**Theorem 2.1:** Let \( (X, P) \) be a partial \( G \)-metric space. Suppose that \( f, g : X \times X \to X \) and \( S, T : X \to X \) be satisfying

\( (2.1.1) \) \( f(X \times X) \subseteq T(X), g(X \times X) \subseteq S(X) \),

\( (2.1.2) \) \( \{f, S\} \) and \( \{g, T\} \) are \( w \)-compatible pairs,

\( (2.1.3) \) One of \( T(X) \) or \( S(X) \) is \( 0-P-G \)-complete subspace of \( X \).
(2.1.4) \( (a) P(f(x, y), f(x, y), g(u, v)) \)

\[
\begin{align*}
P(Sx, Sx, Tu), P(Sy, Sy, Tv), \\
P(f(x, y), f(x, y), Sx), P(f(y, x), f(y, x), Sy), \\
P(g(u, v), g(u, v), Tu), P(g(v, u), g(v, u), Tv),
\end{align*}
\]

\[
\leq k_{\max} \left\{ \begin{array}{l}
\frac{1}{2} \left[ P(f(x, y), f(x, y), Tu) + P(g(u, v), g(u, v), Sx) \right], \\
\frac{1}{2} \left[ P(f(y, x), f(y, x), Tv) + P(g(v, u), g(v, u), Sy) \right]
\end{array} \right.
\]

and

\[
(b) P(f(x, y), g(u, v), g(u, v))
\]

\[
\begin{align*}
P(Sx, Tu, Tu), P(Sy, Tv, Tv), \\
P(f(x, y), Sx, Sx), P(f(y, x), Sy, Sy), \\
P(g(u, v), Tu, Tu), P(g(v, u), Tv, Tv),
\end{align*}
\]

\[
\leq k_{\max} \left\{ \begin{array}{l}
\frac{1}{2} \left[ P(f(x, y), Tu, Tu) + P(g(u, v), Sx, Sx) \right], \\
\frac{1}{2} \left[ P(f(y, x), Tv, Tv) + P(g(v, u), Sy, Sy) \right]
\end{array} \right.
\]

for all \( x, y, u, v \in X \), where \( k \in [0, \frac{1}{2}) \).

Then \( f, g, S \) and \( T \) have a unique common coupled fixed point in \( X \times X \).

**Proof:** Let \((x_0, y_0) \in (X \times X)\). From (2.1.1), we can construct the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) such that

\[
f(x_{2n}, y_{2n}) = Tx_{2n+1} = z_{2n},
\]

\[
f(y_{2n}, x_{2n}) = Ty_{2n+1} = w_{2n},
\]

\[
g(x_{2n+1}, y_{2n+1}) = Sx_{2n+2} = z_{2n+1},
\]

\[ g(y_{2n+1}, x_{2n+1}) = Sy_{2n+2} = w_{2n+1}, \quad n = 0, 1, 2 \]

Now from (2.1.4)(b), we have

\[ P(z_{n+1}, z_{n+1}, z_{2n}) = P(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(x_{2n}, y_{2n})) \]

\[
\leq k \max \left\{ P(z_{2n-1}, z_{2n}, z_{2n}), P(w_{2n-1}, w_{2n}, w_{2n}), \\
2P(z_{2n}, z_{2n}, z_{2n-1}), 2P(w_{2n}, w_{2n}, w_{2n-1}), \\
2P(z_{2n+1}, z_{2n+1}, z_{2n}), 2P(w_{2n+1}, w_{2n+1}, w_{2n}) \\
\right. \]

\[
\leq k \max \left\{ P(z_{2n-1}, z_{2n}, z_{2n}), P(w_{2n-1}, w_{2n}, w_{2n}), \\
2P(z_{2n}, z_{2n}, z_{2n-1}), 2P(w_{2n}, w_{2n}, w_{2n-1}), \\
2P(z_{2n+1}, z_{2n+1}, z_{2n}), 2P(w_{2n+1}, w_{2n+1}, w_{2n}) \\
\right. \]

\[
\left. \frac{1}{2}[2P(z_{2n+1}, z_{2n+1}, z_{2n}) + 2P(z_{2n}, z_{2n}, z_{2n-1})], \\
\frac{1}{2}[2P(w_{2n+1}, w_{2n+1}, w_{2n}) + 2P(w_{2n}, w_{2n}, w_{2n-1})] \right\} \]

from (P_0) and P r o p o s e (in olv)

\[ = 2k \max \left\{ P(z_{2n-1}, z_{2n}, z_{2n}), P(z_{2n+1}, z_{2n+1}, z_{2n}) \right. \]

\[ \left. , P(w_{2n-1}, w_{2n}, w_{2n}), P(w_{2n+1}, w_{2n+1}, w_{2n}) \right\} \quad (2.1) \]

Similarly we can prove,

\[ P(w_{2n+1}, w_{2n+1}, w_{2n}) \leq 2k \max \left\{ P(z_{2n-1}, z_{2n}, z_{2n}), \\
P(z_{2n+1}, z_{2n+1}, z_{2n}), \\
P(w_{2n-1}, w_{2n}, w_{2n}), \\
P(w_{2n+1}, w_{2n+1}, w_{2n}) \right\} \quad (2.2) \]

Thus from (2.1) and (2.2), we have
Now suppose that
\[
\max \{P(z_{2n}, z_{2n}, z_{2n-1}), P(w_{2n}, w_{2n}, w_{2n-1})\} = 0.
\]
Then we have \( z_{2n-1} = z_{2n} \) and \( w_{2n-1} = w_{2n} \) from Proposition 1.1(ii)

From (2.3),
\[
\max \{P(z_{2n-1}, z_{2n-1}, z_{2n}), P(w_{2n-1}, w_{2n-1}, w_{2n})\} = 0 \quad (2.4)
\]
so that \( z_{2n} = z_{2n+1} \) and \( w_{2n} = w_{2n+1} \).

Now from (2.1.4)(a), we can prove
\[
P(z_{2n+2}, z_{2n+2}, z_{2n+1}) \leq 2k \max \left\{ \left\{ P(z_{2n+1}, z_{2n+1}, z_{2n}) , P(z_{2n+2}, z_{2n+2}, z_{2n+1}) \right\} , \left\{ P(w_{2n+1}, w_{2n+1}, w_{2n}) , P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right\} \quad (2.5)
\]
and
\[
P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leq 2k \max \left\{ \left\{ P(z_{2n+1}, z_{2n+1}, z_{2n}) , P(z_{2n+2}, z_{2n+2}, z_{2n+1}) \right\} , \left\{ P(w_{2n+1}, w_{2n+1}, w_{2n}) , P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right\} \quad (2.6)
\]
Thus from (2.5) and (2.6), we have
\[
\max \left\{ \left\{ P(z_{2n+2}, z_{2n+2}, z_{2n+1}), P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right\} \leq 2k \max \left\{ \left\{ P(z_{2n+1}, z_{2n+1}, z_{2n}), P(z_{2n+2}, z_{2n+2}, z_{2n+1}) \right\} , \left\{ P(w_{2n+1}, w_{2n+1}, w_{2n}), P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right\} \quad (2.7)
\]
Using (2.4) in (2.7), we get

\[ \max \{ P(z_{2n+2}, z_{2n+1}, z_{2n+1}), P(w_{2n+2}, w_{2n+2}, w_{2n+1}) \} = 0 \]

so that \( z_{2n+2} = z_{2n+1} \) and \( w_{2n+2} = w_{2n+1} \).

Continuing in this way we get \( z_{2n} = z_{2n+1} = z_{2n+2} = \cdots \) and \( w_{2n} = w_{2n+1} = w_{2n+2} = \cdots \)

Thus \( \{ z_n \} \) and \( \{ w_n \} \) are Cauchy sequences.

Assume that \( \max \{ P(z_{n+1}, z_n, z_n), P(w_{n+1}, w_n, w_n) \} > 0 \) for all \( n \).

Now from (2.3) and (2.7) we have

\[
\max \left\{ \frac{P(z_{n+1}, z_{n+1}, z_n)}{P(w_{n+1}, w_{n+1}, w_n)} \right\} \leq 2k \max \left\{ \frac{P(z_{n-1}, z_n, z_n)}{P(w_{n-1}, w_n, w_n)} \right\} \\
\leq (2k)^2 \max \left\{ \frac{P(z_{n-2}, z_{n-1}, z_{n-1})}{P(w_{n-2}, w_{n-1}, w_{n-1})} \right\} \\
\vdots \\
\leq (2k)^n \max \left\{ \frac{P(z_0, z_1, z_1)}{P(w_0, w_1, w_1)} \right\}
\]

Thus

\[ \lim_{n \to \infty} P(z_n, z_{n+1}, z_{n+1}) = 0 \] \hspace{1cm} (2.8)

and

\[ \lim_{n \to \infty} P(w_n, w_{n+1}, w_{n+1}) = 0 \] \hspace{1cm} (2.9)

For \( m, n \in \mathbb{N} \) with \( m > n \), we have

\[ P(z_n, z_m, z_m) \leq P(z_n, z_{n+1}, z_{n+1}) + P(z_{n+1}, z_{n+2}, z_{n+2}) + \cdots + P(z_{m-1}, z_m, z_m) \]
\[
\leq [(2k)^n + (2k)^{n+1} + \ldots + (2k)^{m-1}] \max \left\{ \frac{P(z_0, z_1, z_i)}{P(w_0, w_1, w_i)} \right\} \\
\leq \frac{(2k)^n}{1-2k} \max \left\{ \frac{P(z_0, z_1, z_i)}{P(w_0, w_1, w_i)} \right\}
\]

Thus

\[
\lim_{n,m \to \infty} P(z_n, z_m, z_m) = 0 \quad \text{(2.10)}
\]

Similarly, we have

\[
\lim_{n,m \to \infty} P(w_n, w_m, w_m) = 0 \quad \text{(2.11)}
\]

Thus \{z_n\} and \{w_n\} are 0-P-G- Cauchy sequences in \(X\).

Suppose \(S(X)\) is 0-P-G complete. Then the sequences \(\{z_{2n+1}\} = \{Sx_{2n+2}\}\) and \(\{w_{2n+1}\} = \{Sy_{2n+2}\}\) P-G converge to points \(\alpha, \beta \in S(X)\) such that \(P(\alpha, \alpha, \alpha) = 0\) and \(P(\beta, \beta, \beta) = 0\) and \(\alpha = Su\) and \(\beta = Sv\) for some \(u, v \in X\).

Since \{z_n\} and \{w_n\} are 0-P-G- Cauchy and from (2.8) and (2.9), it follows that \{z_{2n}\} and \{w_{2n}\} are P-G- converge to \(\alpha\) and \(\beta\) respectively.

Using (2.1.4)(b), we obtain that

\[
P(z_{2n+1}, z_{2n+1}, f(u, v)) = P(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(u, v))
\]

\[
\leq k \max \left\{ \frac{1}{2} [P(f(u, v), z_{2n}, z_{2n}) + P(z_{2n+1}, Su, Su)], \frac{1}{2} [P(f(v, u), w_{2n}, w_{2n}) + P(w_{2n+1}, Sv, Sv)] \right\}
\]

Letting \(n \to \infty\) we have
Since \( f(u) = u \), we get
\[
\alpha + \beta = \beta.
\]
Thus
\[
f_S = \{ (\alpha, \alpha), P(f(v, u), \beta, \beta) \}
\]
from (2.8), (2.9), Lemmas (1.1) and (1.2)
\[
= k \max \{ P(f(u, v), \alpha, \alpha), P(f(v, u), \beta, \beta) \}
\]
Using (2.1.4)(b) to \( P(w_{2n+1}, w_{2n+1}, f(v, u)) \) and then letting \( n \to \infty \), we get
\[
P(\beta, \beta, f(v, u)) \leq k \max \{ P(f(u, v), \alpha, \alpha), P(f(v, u), \beta, \beta) \}
\]
Thus from (2.12) and (2.13) we have
\[
\max \left\{ P(\alpha, \alpha, f(u, v)) \right\} \leq k \max \{ P(f(u, v), \alpha, \alpha), P(f(v, u), \beta, \beta) \}
\]
which in turn yields from Proposition 1.1(ii) that \( f(u, v) = \alpha = Su \) and \( f(v, u) = \beta = Sv \). Thus \((\alpha, \beta)\) is a coupled coincidence point of \( f \) and \( S \). Since \( \{f, S\} \) is a \( w \)-compatible pair, we have \( S\alpha = f(\alpha, \beta) \) and \( S\beta = f(\beta, \alpha) \).

We next prove that \( S\alpha = \alpha \) and \( S\beta = \beta \).

Applying (2.1.4)(b), we obtain that
\[
P(z_{2n+1}, z_{2n+1}) = P(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(\alpha, \beta))
\]
\[
\leq k \max \left\{ \begin{array}{l}
P(S\alpha, z_{2n}, z_{2n}), P(S\beta, w_{2n}, w_{2n}), \\
P(f(\alpha, \beta), S\alpha, S\alpha), P(f(\beta, \alpha), S\beta, S\beta), \\
P(z_{2n+1}, z_{2n}, z_{2n}), P(w_{2n+1}, w_{2n}, w_{2n}) \\
\end{array} \right\}
\]
\[
\leq k \max \left\{ \begin{array}{l}
\frac{1}{2}[P(f(\alpha, \beta), z_{2n}, z_{2n}) + P(z_{2n+1}, S\alpha, S\alpha)], \\
\frac{1}{2}[P(f(\beta, \alpha), w_{2n}, w_{2n}) + P(w_{2n+1}, S\beta, S\beta)] \\
\end{array} \right\}
\]
Taking $n \to \infty$, we have

$$P(\alpha, \alpha, S\alpha) \leq k \max \left\{ \begin{array}{l}
P(\alpha, \alpha, \alpha), P(S\beta, \beta, \beta), \\
P(\alpha, S\alpha, S\alpha), P(S\beta, S\beta, S\beta), 0,0,
\end{array} \right\}$$

from lemma (1.1), (1.2) and (2.8), (2.9)

$$\leq k \max \left\{ \begin{array}{l} P(\alpha, \alpha, \alpha) + P(\alpha, S\alpha, S\alpha), \\
P(S\beta, \beta, \beta) + P(\beta, S\beta, S\beta), \\
\end{array} \right\}$$

(2.14)

Using (2.1.4)(b) to $P(w_{2n+1}, w_{2n+1}, S\beta)$ and then letting $n \to \infty$, we get

$$P(\beta, \beta, S\beta) \leq k \max \left\{ \begin{array}{l}
P(\alpha, \alpha, \alpha) + P(\alpha, S\alpha, S\alpha), \\
P(S\beta, \beta, \beta) + P(\beta, S\beta, S\beta),
\end{array} \right\}$$

(2.15)

From (2.14) and (2.15), we have

$$\max \left\{ \begin{array}{l}
P(\alpha, S\alpha), \\
P(\beta, S\beta)
\end{array} \right\} \leq k \max \left\{ \begin{array}{l} P(\alpha, \alpha, \alpha) + P(\alpha, S\alpha, S\alpha), \\
P(S\beta, \beta, \beta) + P(\beta, S\beta, S\beta),
\end{array} \right\}$$

$$\leq k \left[ \max \{P(\alpha, \alpha, \alpha), P(S\beta, \beta, \beta)\} \\
+ \max \{P(\alpha, S\alpha, S\alpha), P(\beta, S\beta, S\beta)\} \right]$$. 
Thus

\[
\max \left\{ \frac{P(\alpha, \alpha, S\alpha)}{P(\beta, \beta, S\beta)} \right\} \leq \frac{k}{1 - k} \max \left\{ \frac{P(\alpha, S\alpha, S\alpha)}{P(\beta, S\beta, S\beta)} \right\}
\]

(2.16)

Using (2.1.4)(a), we have

\[
P(z_{2n+1}, S\alpha, S\alpha) = P(g(x_{2n+1}, y_{2n+1}), f(\alpha, \beta), f(\alpha, \beta)) \leq k \max \left\{ \frac{P(S\alpha, S\alpha, z_{2n})}{P(S\alpha, S\alpha, S\alpha)} + P(S\beta, S\beta, w_{2n}), \right.\]

\[
\frac{1}{2} \left[ P(S\alpha, S\alpha, z_{2n}) + P(z_{2n+1}, z_{2n+1}, S\alpha) \right],
\]

\[
\frac{1}{2} \left[ P(S\beta, S\beta, w_{2n}) + P(w_{2n+1}, w_{2n+1}, S\beta) \right]
\]

Taking \( n \to \infty \), we have

\[
P(\alpha, S\alpha, S\alpha) \leq k \max \left\{ \frac{P(S\alpha, S\alpha, S\alpha)}{P(\beta, \beta, S\beta, \beta)} \right.\]

\[
\frac{1}{2} \left[ P(S\alpha, S\alpha, S\alpha) + P(\alpha, \alpha, S\alpha) \right],
\]

\[
\frac{1}{2} \left[ P(S\beta, S\beta, S\beta) + P(\beta, \beta, S\beta) \right]
\]

\[= k \max \left\{ \frac{P(S\alpha, S\alpha, S\alpha) + P(\alpha, S\alpha, S\alpha)}{P(S\beta, \beta, S\beta)} \right\}
\]

(2.17)

Applying (2.1.4)(a) to \( P(w_{2n+1}, S\beta, S\beta) \) and then letting \( n \to \infty \), we get
From (2.17) and (2.18), we have

\[
\max \left\{ \frac{P(\alpha, S\alpha, S\alpha)}{P(\beta, S\beta, S\beta)} \right\} \leq k \max \left\{ \frac{P(S\alpha, \alpha, \alpha) + P(\alpha, S\alpha, S\alpha)}{P(S\beta, \beta, \beta) + P(\beta, S\beta, S\beta)} \right\}
\]

Thus

\[
\max \left\{ \frac{P(\alpha, S\alpha, S\alpha)}{P(\beta, S\beta, S\beta)} \right\} \leq k \left( \max \left\{ \frac{P(S\alpha, \alpha, \alpha)}{P(S\beta, \beta, \beta)} \right\} \right)
\]

From (2.16) and (2.19), we have

\[
\max \left\{ \frac{P(\alpha, S\alpha, S\alpha)}{P(\beta, S\beta, S\beta)} \right\} \leq \left( \frac{k}{1-k} \right)^2 \max \left\{ \frac{P(S\alpha, \alpha, \alpha)}{P(S\beta, \beta, \beta)} \right\}
\]

so that \( S\alpha = \alpha \) and \( S\beta = \beta \). Thus \( \alpha = S\alpha = f(\alpha, \beta) \) and \( \beta = S\beta = f(\beta, \alpha) \). Since \( f(X \times X) \subseteq T(X) \), there exist \( a, b \in X \) such that \( \alpha = f(\alpha, \beta) = Ta \) and \( \beta = f(\beta, \alpha) = Tb \).

From (2.1.4)(a) we obtain

\[
P(\alpha, \alpha, g(a, b)) = P(f(\alpha, \beta), f(\alpha, \beta), g(a, b))
\]

\[
\leq k \max \left\{ \frac{P(\alpha, \alpha, \alpha) + P(\beta, \beta, \beta)}{P(\alpha, \alpha, \alpha) + P(\beta, \beta, \beta)}, \frac{1}{2} [P(\alpha, \alpha, \alpha) + P(g(a, b), g(a, b), \alpha)], \frac{1}{2} [P(\beta, \beta, \beta) + P(g(b, a), g(b, a), \beta)] \right\}
\]
\[
\leq 2k \max \left\{ \frac{P(g(a,b), \alpha, \alpha)}{P(g(b,a), \beta, \beta)} \right\} \quad (2.20)
\]

Again using (2.1.4)(a) to \( P(\beta, \beta, g(b,a)) \), we obtain

\[
P(\beta, \beta, g(b,a)) \leq 2k \max \left\{ \frac{P(g(a,b), \alpha, \alpha)}{P(g(b,a), \beta, \beta)} \right\} \quad (2.21)
\]

From (2.20) and (2.21), we have

\[
\max \left\{ \frac{P(\alpha, \alpha, g(a,b))}{P(\alpha, \alpha, g(a,b))} \right\} \leq k \max \left\{ \frac{P(\beta, \beta, g(b,a))}{P(\alpha, \alpha, g(a,b))} \right\}
\]

so that \( g(a,b) = \alpha = T\alpha \) and \( g(b,a) = \beta = T\beta \). Since the pair \{\(g,T\}\} is weakly compatible, we have \( T\alpha = g(\alpha, \beta) \) and \( T\beta = g(\beta, \alpha) \). Now we prove \( T\alpha = \alpha \) and \( T\beta = \beta \).

Using (2.1.4)(a) we obtain

\[
P(\alpha, \alpha, T\alpha) = P(f(\alpha, \beta), f(\alpha, \beta), g(\alpha, \beta))
\]

\[
\leq k \max \left\{ \frac{P(\alpha, \alpha, T\alpha), P(\beta, \beta, T\beta)}{P(\alpha, \alpha, P(\beta, \beta)}, P(\alpha, \alpha, P(\alpha, \beta, \beta)}, P(T\alpha, T\alpha, T\alpha), P(T\beta, T\beta, T\beta)}, \right. \]

\[
\left. \frac{1}{2} [P(\alpha, \alpha, T\alpha) + P(T\alpha, T\alpha, \alpha)], \frac{1}{2} [P(\beta, \beta, T\beta) + P(T\beta, T\beta, \beta)] \right\}
\]

\[
\leq k \max \left\{ \frac{P(\alpha, \alpha, T\alpha), P(\beta, \beta, T\beta), 0, 0, P(T\alpha, T\alpha, T\alpha) + P(\alpha, T\alpha, \alpha)}, P(\beta, \beta, T\beta) + P(\beta, T\beta, T\beta)}, \right. \]

\[
\left. \frac{1}{2} [P(\alpha, \alpha, T\alpha) + P(T\alpha, T\alpha, \alpha)], \frac{1}{2} [P(\beta, \beta, T\beta) + P(T\beta, T\beta, \beta)] \right\}
\]
\[
= k \max \left\{ \frac{P(T\alpha, \alpha, \alpha) + P(\alpha, T\alpha, T\alpha)}{P(T\beta, \beta, \beta) + P(\beta, T\beta, T\beta)} \right\}
\] (2.22)

Similarly from (2.1.4)(a), we obtain
\[
P(\beta, \beta, T\beta) \leq k \max \left\{ \frac{P(T\alpha, \alpha, \alpha) + P(\alpha, T\alpha, T\alpha)}{P(T\beta, \beta, \beta) + P(\beta, T\beta, T\beta)} \right\}
\] (2.23)

From (2.22) and (2.23)
\[
\max \left\{ \frac{P(\alpha, \alpha, T\alpha)}{P(\beta, \beta, T\beta)} \right\} \leq k \max \left\{ \frac{P(T\alpha, \alpha, \alpha) + P(\alpha, T\alpha, T\alpha)}{P(T\beta, \beta, \beta) + P(\beta, T\beta, T\beta)} \right\}
\]

Thus
\[
\max \left\{ \frac{P(\alpha, \alpha, T\alpha)}{P(\beta, \beta, T\beta)} \right\} \leq \frac{k}{1-k} \frac{P(\alpha, \alpha, T\alpha)}{P(\beta, \beta, T\beta)}
\] (2.24)

Now using (2.1.4)(b) as in above, we obtain
\[
\max \left\{ \frac{P(T\alpha, T\alpha, \alpha)}{P(T\beta, T\beta, \beta)} \right\} \leq \frac{k}{1-k} \frac{P(T\alpha, T\alpha, \alpha)}{P(T\beta, T\beta, \beta)}
\] (2.25)

From (2.24) and (2.25), we have
\[
\max \left\{ \frac{P(\alpha, \alpha, T\alpha)}{P(\beta, \beta, T\beta)} \right\} \leq \frac{k^2}{(1-k)^2} \frac{P(\alpha, \alpha, T\alpha)}{P(\beta, \beta, T\beta)}
\]

so that \( \alpha = T\alpha \) and \( \beta = T\beta \). Thus \( \alpha = T\alpha = g(\alpha, \beta) \) and \( \beta = T\beta = g(\beta, \alpha) \). Hence \( (\alpha, \beta) \) is a common coupled fixed point of \( f, g, S \) and \( T \).

Suppose that \( (\alpha^l, \beta^l) \in X \times X \) is another common coupled fixed point of \( f, g, S \) and \( T \).
Suppose that $\alpha \neq \alpha'$ and $\beta \neq \beta'$.

Applying (2.1.4)(a), we obtain that

$$P(\alpha, \alpha, \alpha') = P(f(\alpha, \beta), f(\alpha, \beta), g(\alpha', \beta'))$$

\[
\leq k \max \left\{ \begin{array}{c}
P(\alpha, \alpha, \alpha'), P(\beta, \beta, \beta'), \\
P(\alpha, \alpha, \alpha), P(\beta, \beta, \beta), \\
P(\alpha', \alpha', \alpha'), P(\beta', \beta', \beta'), \\
\frac{1}{2} \left[ P(\alpha, \alpha, \alpha') + P(\alpha', \alpha', \alpha) \right], \\
\frac{1}{2} \left[ P(\beta, \beta, \beta') + P(\beta', \beta', \beta) \right]
\end{array} \right\}
\]

\[
\leq k \max \left\{ \begin{array}{c}
P(\alpha, \alpha, \alpha'), P(\beta, \beta, \beta'), 0, 0, \\
P(\alpha', \alpha, \alpha) + P(\alpha, \alpha', \alpha'), \\
P(\beta', \beta, \beta) + P(\beta, \beta', \beta'), \\
\frac{1}{2} \left[ P(\alpha, \alpha, \alpha') + P(\alpha', \alpha', \alpha) \right], \\
\frac{1}{2} \left[ P(\beta, \beta, \beta') + P(\beta', \beta', \beta) \right]
\end{array} \right\}
\]

from (P_6)

\[
= k \max \left\{ P(\alpha', \alpha, \alpha) + P(\alpha, \alpha', \alpha'), \\
P(\beta', \beta, \beta) + P(\beta, \beta', \beta') \right\}
\]

(2.26)

Again using (2.1.4)(a), we obtain

$$P(\beta, \beta, \beta') \leq k \max \left\{ \begin{array}{c}
P(\alpha', \alpha, \alpha) + P(\alpha, \alpha', \alpha'), \\
P(\beta', \beta, \beta) + P(\beta, \beta', \beta')
\end{array} \right\}
\]

(2.27)

From (2.26) and (2.27), we have
So that
\[
\max \left\{ P(\alpha, \alpha, \alpha^1), \right\} \leq k \max \left\{ P(\alpha^1, \alpha, \alpha) + P(\alpha, \alpha^1, \alpha), \right\} \\
\leq k \left[ \max \{P(\alpha^1, \alpha, \alpha), P(\beta^1, \beta, \beta)\} \ight] \\
\leq \left[ \max \{P(\alpha, \alpha, \alpha), P(\beta, \beta, \beta)\} \right] \\
+ \max \{P(\alpha, \alpha^1, \alpha), P(\beta, \beta^1, \beta)\}
\]
so that
\[
\max \left\{ P(\alpha, \alpha, \alpha^1), \right\} \leq \frac{k}{1-k} \max \left\{ P(\alpha, \alpha^1, \alpha^1) \right\} \\
\leq \frac{k}{1-k} \max \left\{ P(\alpha, \alpha, \alpha), \right\}
\]
(2.28)

Similarly applying (2.1.4)\((b)\) to \(P(\alpha^1, \alpha^1, \alpha)\) and \(P(\beta^1, \beta^1, \beta)\), we obtain that
\[
\max \left\{ P(\alpha^1, \alpha^1, \alpha), \right\} \leq \frac{k}{1-k} \max \left\{ P(\alpha, \alpha, \alpha^1) \right\}
\]
(2.29)

From (2.28) and (2.29), we have
\[
\max \left\{ P(\alpha, \alpha, \alpha^1), \right\} \leq \left( \frac{k}{1-k} \right)^2 \max \left\{ P(\alpha, \alpha, \alpha^1) \right\}
\]
(2.30)
so that \(\alpha = \alpha^1\) and \(\beta = \beta^1\). Thus \((\alpha, \beta)\) is the unique common coupled fixed point of \(f, g, S\) and \(T\).

If we put \(f = g\) and \(S = T\) in Theorem 2.1, we have the following Corollary.

**Corollary 2.1** Let \((X, P)\) be a partial \(G\)-metric space. Suppose that \(f : X \times X \rightarrow X\) and \(S : X \rightarrow X\) be satisfying

(2.1.1) \(f(X \times X) \subseteq T(X)\),

(2.1.2) \((f, T)\) are weakly compatible pairs,

(2.1.3) \(T(X)\) is \(0 - P - G\)-complete subspace of \(X\),

(2.1.4) \((a) P(f(x, y), f(x, y), f(u, v))\)


\[
\begin{align*}
\leq k_{\max} \left\{ & P(Tx, Tx, Tu), P(Ty, Ty, Tv), \\
& P(f(x, y), f(x, y), Tx), P(f(y, x), f(y, x), Ty), \\
& P(f(u, v), f(u, v), Tu), P(f(v, u), f(v, u), Tv), \\
& \frac{1}{2} [P(f(x, y), f(x, y), Tu) + P(f(u, v), f(u, v), Tx)], \\
& \frac{1}{2} [P(f(y, x), f(y, x), Tv) + P(f(v, u), f(v, u), Ty)]
\right\}
\end{align*}
\]

and

\[
\begin{align*}
\leq k_{\max} \left\{ & P(Tx, Tu, Tu), P(Ty, Ty, Tv), \\
& P(f(x, y), Tx, Tx), P(f(y, x), Ty, Ty), \\
& P(f(u, v), Tu, Tu), P(f(v, u), Tv, Tv), \\
& \frac{1}{2} [P(f(x, y), Tu, Tu) + P(f(u, v), Tx, Tx)], \\
& \frac{1}{2} [P(f(y, x), Tv, Tv) + P(f(v, u), Ty, Ty)]
\right\}
\end{align*}
\]

for all \( x, y, u, v \in X \), where \( k \in \left[ 0, \frac{1}{2} \right) \).

Then \( f \) and \( T \) have a unique common coupled fixed point in \( X \times X \).

Now we give the following example to illustrate our Theorem 2.1.

**Example 2.1** Let \((X, P)\) be a partial \(G\)-metric space, where \( X = [0,1] \) \( P : X \times X \times X \to [0, \infty) \) be defined by

\[
P(x, y, z) = \max\{x, y\} + \max\{y, z\} + \max\{x, z\}.
\]

Let \( f, g : X \times X \to X \) and \( S, T : X \to X \) be defined by

\[
f(x, y) = \frac{x^2 + y^2}{16}, \quad g(x, y) = \frac{x + y}{32}, \quad Sx = \frac{x^2}{2}, \quad Tx = \frac{x}{4}, \forall x, y \in X.
\]
The conditions (2.1.1), (2.1.2) and (2.1.3) are obvious.

For all \( x, y \in X \), consider

\[
P(f(x, y), f(x, y), g(u, v)) = f(x, y) + 2 \max \{ f(x, y), g(u, v) \}
\]

\[
= \frac{x^2 + y^2}{16} + 2 \max \left\{ \frac{x^2 + y^2}{16}, \frac{u + v}{32} \right\}
\]

\[
= \left[ \frac{x^2}{16} + 2 \max \left\{ \frac{x^2}{16}, \frac{u}{32} \right\} \right] + \left[ \frac{y^2}{16} + 2 \max \left\{ \frac{y^2}{16}, \frac{v}{32} \right\} \right]
\]

\[
= \frac{1}{8} \left[ \frac{x^2}{2} + 2 \max \left\{ \frac{x^2}{2}, \frac{u}{4} \right\} \right] + \frac{1}{8} \left[ \frac{y^2}{2} + 2 \max \left\{ \frac{y^2}{2}, \frac{v}{4} \right\} \right]
\]

\[
= \frac{1}{8} \left[ Sx + 2 \max \{ Sx, Ty \} \right] + \frac{1}{4} \left[ Sx + 2 \max \{ Sx, Ty \} \right]
\]

\[
= \frac{1}{8} \left[ P(Sx, Sx, Tu) + P(Sy, Sy, Tv) \right]
\]

\[
= \frac{1}{8} \left[ P(Sx, Sx, Tu) + P(Sy, Sy, Tv) \right]
\]

\[
= \frac{1}{8} \left[ P(Sx, Sx, Tu) + P(Sy, Sy, Tv) \right]
\]

\[
\leq \frac{1}{4} \max \{ P(Sx, Sx, Tu), P(Sy, Sy, Tv) \}
\]

\[
\leq \frac{1}{4} \max \left\{ P(Sx, Sx, Tu, P(Sy, Sy, Tv),
\right. \]

\[
\left. P(f(x, y), f(x, y), Sx) \right) + P(f(y, x), f(y, x), Sy),
\right. \]

\[
P(g(u, v), g(u, v), Tu), P(g(v, u), g(v, u), Tv),
\right. \]

\[
\left. \frac{1}{2} [P(f(x, y), f(x, y), Tu) + P(g(u, v), g(u, v), Sx)],
\right. \]

\[
\left. \frac{1}{2} [P(f(y, x), f(y, x), Tv) + P(g(v, u), g(v, u), Sy)] \right\}
\]

One can easily verify (2.1.4)(b) in the similar lines.
Thus all conditions of Theorem (2.1) are satisfied. Clearly (0,0) is the unique common coupled fixed point of $f, g, S$ and $T$.

REFERENCES


