POWERS $x^n$ IN TERMS OF MODIFIED CHEBYSHEV POLYNOMIALS

1J. López-Bonilla*, 1S. Barragán-Gómez, 2Bhadraman Tuladhar

1ESIME-Zacatenco, Instituto Politécnico Nacional, Anexo Edif. 3, Col. Lindavista, CP 07738 México D.F.

2Department of Natural Sciences (Mathematics), School of Science, Kathmandu University, P.O. Box 6250, Kathmandu, Nepal

*Corresponding author: jlopezb@ipn.mx

Received 25 January, 2009; Revised 11 May, 2010

ABSTRACT
We exhibit two procedures to express $x^n$ in terms of the shifted Chebyshev polynomials, which is useful to reduce the degree of a polynomial in the interval $[0,1]$.

Keywords: Chebyshev-Lanczos polynomials

INTRODUCTION
In numerical analysis may be necessary to reduce, with small error, the degree of a polynomial in the interval $[0, 1]$, which is possible employing the Modified Chebyshev polynomials $\bar{T}_r(x)$ defined by [1]:

$$\bar{T}_0(x) = \frac{1}{2}, \quad \bar{T}_k(x) = T_k(2x - 1), \quad k = 1,2,...$$

where the first-kind Chebyshev polynomials $T_r(x)$ are given by the recurrence relation [2-6]:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1,2,...$$

therefore

$$\bar{T}_0(x) = \frac{1}{2}, \quad \bar{T}_1(x) = 2x-1, \quad \bar{T}_2(x) = 8x^2 - 8x + 1, \quad \bar{T}_3(x) = 32x^3 - 48x^2 + 18x - 1, \quad \bar{T}_4(x) = 128x^4 - 256x^3 - 32x + 1, \quad \text{etc.}$$
In the mentioned reduction process we need the powers \( x^n \) in terms of \( \bar{T}_r \), then from (3):

\[
x^0 = 2\bar{T}_0, \quad x = \frac{1}{2}(2\bar{T}_0 + \bar{T}_1), \quad x^2 = \frac{1}{8}(6\bar{T}_0 + 4\bar{T}_1 + \bar{T}_2),
\]

\[
x^3 = \frac{1}{32}(20\bar{T}_0 + 15\bar{T}_1 + 6\bar{T}_2 + \bar{T}_3), \quad x^4 = \frac{1}{128}(70\bar{T}_0 + 56\bar{T}_1 + 28\bar{T}_2 + 8\bar{T}_3 + \bar{T}_4), \quad \text{etc.}
\]

that is [1]:

\[
\frac{1}{2} (4x)^n = \sum_{r=0}^{n} \binom{2n}{n-r} \bar{T}_r, \quad n = 0, 1, \ldots
\]  

(5)

The next section exhibits an algorithm to obtain \( x^j \) in function of \( \bar{T}_r \) if we know the corresponding expansion of \( x^{j-1} \), and also another procedure which employs to (5) as a Newton’s binomial expression.

\( x^n \) in terms of \( \bar{T}_r \)

We may write (5) in the form:

\[
\begin{array}{cccccc}
T_0 & T_1 & T_2 & T_3 & T_4 & \cdots \\
\frac{1}{2} (4x)^0 & 1 & 0 & 0 & 0 & 0 \cdots \\
\frac{1}{2} (4x)^1 & 2 & 1 & 0 & 0 & 0 \cdots \\
\frac{1}{2} (4x)^2 & 6 & 4 & 1 & 0 & 0 \cdots \\
\frac{1}{2} (4x)^3 & 20 & 15 & 6 & 1 & 0 \cdots \\
\frac{1}{2} (4x)^4 & 70 & 56 & 28 & 8 & 1 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]  

(6)

or in function of the columns vectors \( \frac{1}{2} (4x)^j \) and \( (\bar{T}_r) \) for a given \( n \):

\[
\begin{pmatrix}
\frac{1}{2} (4x)^0 \\
\frac{1}{2} (4x)^1 \\
\vdots \\
\frac{1}{2} (4x)^n
\end{pmatrix}
= A
\begin{pmatrix}
\bar{T}_0 \\
\bar{T}_1 \\
\vdots \\
\bar{T}_n
\end{pmatrix}
\]  

(7)
where $A$ is the $(n + 1) \times (n + 1)$ triangular matrix of coefficients appearing in (6):

$$A = (a_{jr}) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & \cdots \\
6 & 4 & 1 & 0 & \cdots \\
20 & 15 & 6 & 1 & \cdots \\
70 & 56 & 28 & 8 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad j, r = 0, 1, \ldots, n \tag{8}
$$

then $(\bar{T}_r) = A^{-1} \cdot \left(\frac{1}{2} (4x)' \right)$ reproduces (3).

The relations (5) and (7) imply that:

$$a_{jr} = \begin{pmatrix} 2n \\ n - r \end{pmatrix}, \quad j, r = 0, 1, \ldots \tag{9}
$$

thus

$$a_{j0} = 1, \quad a_{jr} = 0, \quad r > j \tag{10}
$$

and we can prove the following properties not found explicitly in the literature:

$$a_{j+1,0} = 2(a_{j0} + a_{j1}), \quad j = 0, 1, 2, \ldots$$

$$a_{jr} = a_{j-1,r-1} + 2a_{j-1,r} + a_{j-1,r+1}, \quad r, j = 1, 2, 3, \ldots \tag{11}
$$

The formulae (11) permit to construct the row $j$ of $A$ if we know its row $j-1$, and they represent an algorithm to express $x^n$ in terms of $(\bar{T}_r)$ whose systematic use minimize the amount of arithmetical computations involved in (5).

On the other hand, the expansion (5) can be written as:

$$\frac{1}{2} (4x)' = \sum_{k=0}^{n} \binom{2n}{k} \bar{T}_{n-k} = \sum_{k=0}^{n} \binom{2n}{k} \bar{T}^{n-k} \tag{12}
$$

where we use the notation:

$$\bar{T}^{-j} = 0, \quad j = 1, 2, \ldots, \quad \bar{T}^r \equiv \bar{T}_r, \quad r = 0, 1, 2, \ldots \tag{13}
$$

very employed in Gregory-Newton and Stirling interpolations [7].
Thus (12) adopts the form of a Newton’s binomial expression:

\[
\frac{1}{2} (4x)^n = \frac{1}{T^n} \sum_{k=0}^{2n} \binom{2n}{k} T^{2n-k} = \frac{1}{T^n} (1 + T)^{2n}
\]

which is a procedure alternative to (11) to obtain \( x^n \) in function of \( T \). For example:

\[
\frac{1}{2} (4x)^2 = \frac{1}{T^2} (1 + T)^4 = \frac{1}{T^2} (1 + 4T + 6T^2 + 4T^3 + T^4),
\]

\[
= T^{-2} + 4T^{-1} + 6T^0 + 4T + T^2 = 6T_0 + 4T_1 + T_2, \quad \text{etc.}
\]

in according with (6). The relation (14) may be easily manipulated by a computer via some symbolic language as MAPLE.

**REFERENCES**


