A COMMON FIXED POINT THEOREM FOR SIX EXPANSIVE MAPPINGS IN G – METRIC SPACES

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ABSTRACT:
In this paper we obtain a unique common fixed point theorem for six expansive mappings in G –metric spaces.


Key words : Expansive mappings, G –metric space, Weakly compatible mappings.

1. INTRODUCTION
Dhage [2, 3, 4, 5]. et al. introduced the concept of D –metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [6] and Naidu et al. [10, 11, 12] demonstrated that most of the claims concerning the fundamental topological structure of D – metric space are incorrect, alternatively, Mustafa and Sims introduced in [6] more appropriate notion of generalized metric space which called G – metric spaces, and obtained some topological properties. Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh[7], Mustafa, Shatanawi and Bataineh [8], Mustafa and Sims [9] Shatanawi [13] and Renu Chugh, Tamanna Kadian, Anju Rani and B.E. Rhoades [1] et al. obtained some fixed point theorems for a single map in G- metric spaces. In this paper, we obtain a unique common fixed point theorem for six weakly compatible expansive mappings in G – metric spaces. First, we present some known definitions and propositions in G – metric spaces.

DEFINITION 1.1 [6] : Let X be a nonempty set and let G: X × X × X → R+ be a function satisfying the following properties :

(G1) : G (x, y, z ) = 0 if x = y = z ,
(G_2) : 0 < G (x, x, y) for all x, y \in X with x \neq y,

(G_3) : G (x, x, y) \leq G (x, y, z) for all x, y, z \in X with y \neq z,

(G_4) : G (x, y, z) = G (x, z, y) = G (y, z, x) = \ldots, symmetry in all three variables,

(G_5) : G (x, y, z) \leq G (x, a, a) + G (a, y, z) for all x, y, z, a \in X.

Then the function G is called a generalized metric or a G–metric on X and the pair (X, G) is called a G-metric space.

**Definition 1.2 [6]**: Let (X, G) be a G-metric space and \{x_n\} be a sequence in X. A point x \in X is said to be limit of \{x_n\} iff \(\lim_{n, m \to \infty} G(x_n, x_m, x_N) = 0\). In this case, the sequence \{x_n\} is said to be G–convergent to x.

**Definition 1.3 [6]**: Let (X, G) be a G-metric space and \{x_n\} be a sequence in X. \{x_n\} is called G-Cauchy iff \(\lim_{n, m \to \infty} G(x_n, x_m, x_N) = 0\). (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

**Proposition 1.4 [6]**: In a G-metric space (X, G), the following are equivalent.

1. The sequence \{x_n\} is G-Cauchy.
2. For every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(G (x_n, x_m, x_N) < \varepsilon\) for all \(n, m \geq N\).

**Proposition 1.5 [6]**: Let (X, G) be a G-metric space. Then the function \(G (x, y, z)\) is jointly continuous in all three of its variables.

**Proposition 1.6 [6]**: Let (X, G) be a G-metric space. Then for any \(x, y, z, a \in X\), it follows that

1. if \(G(x, y, z) = 0\) then \(x = y = z\),
2. \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\),
3. \(G(x, y, y) \leq 2G(x, x, y)\),
4. \(G(x, y, z) \leq G(x, a, z) + G(a, y, z)\),
5. \(G(x, y, z) \leq \frac{2}{3} [G(x, a, a) + G(y, a, a) + G(z, a, a)]\).

**Proposition 1.7 [6]**: Let (X, G) be a G-metric space. Then for a sequence \(\{x_n\} \subseteq X\) and a point \(x \in X\), the following are equivalent

1. \(\{x_n\}\) is G-convergent to \(x\),
(ii) \( G(x_n, x_n, x) \to 0 \) as \( n \to \infty \),

(iii) \( G(x_n, x, x) \to 0 \) as \( n \to \infty \),

(iv) \( G(x_m, x_n, x) \to 0 \) as \( m, n \to \infty \).

2. RESULTS

THEOREM 2.1: Let \((X, G)\) be a complete \(G\)-metric space and 
\(S, T, R, f, g, h : X \to X\) be mappings such that

\[
(2.1.1) \quad G(Sx, Ty, Rz) \geq q \max \left\{ G(fx, gy, hz), G(fx, Sx, Rz), \right. \\
\left. \quad G(gy, Ty, Sx), G(hz, Rz, Ty) \right\}
\]

for all \(x, y, z \in X\) and \(q > 1\),

\[
(2.1.2) \quad h(X) \subseteq S(X), f(X) \subseteq T(X), g(X) \subseteq R(X),
\]

\[
(2.1.3) \quad \text{one of } f(X), g(X) \text{ and } h(X) \text{ is a } G\text{-complete subspace of } X,
\]

\[
(2.1.4) \quad \text{the pairs } (f, S), (g, T) \text{ and } (h, R) \text{ are weakly compatible.}
\]

Then (a) one of the pairs \((f, S), (g, T)\) and \((h, R)\) has a coincidence point in \(X\) or

(b) \(S, T, R, f, g\) and \(h\) have a unique common fixed point in \(X\).

PROOF: Let \(x_0 \in X\).

From (2.1.2), there exist \(x_1, x_2, x_3 \in X\) such that \(hx_0 = Sx_1 = y_1\), say,
\(fx_1 = Tx_2 = y_2\), say and \(gx_2 = Rx_3 = y_3\), say.

By induction, there exist sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\(hx_{3n} = Sx_{3n+1} = y_{3n+1}, \ fx_{3n+1} = Tx_{3n+2} = y_{3n+2}, \ gx_{3n+2} = Rx_{3n+3} = y_{3n+3}, \ n = 0, 1, 2, \ldots\)

If \(y_{3n+1} = y_{3n+2}\) then \(f x = S x\), where \(x = x_{3n+1}\).

If \(y_{3n+2} = y_{3n+3}\) then \(g x = T x\), where \(x = x_{3n+2}\).

If \(y_{3n} = y_{3n+1}\) then \(h x = R x\), where \(x = x_{3n}\).

Assume that \(y_n \neq y_{n+1}\) for all \(n\).

Denote \(d_n = G(y_n, y_{n+1}, y_{n+2})\).

\[
d_{3n-1} = G(y_{3n-1}, y_{3n}, y_{3n+1}) \\
= G(Tx_{3n-1}, Rx_{3n}, Sx_{3n+1})
\]
\[ \geq q \max \left\{ G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n+2}, y_{3n+1}, y_{3n}) \right\} \]

\[ = q \max \{ d_{3n}, d_{3n}, d_{3n-1}, d_{3n-1} \} . \]

Thus we have \( d_{3n} \geq q d_{3n} \) so that \( d_{3n} \leq k d_{3n-1} \), where \( k = \frac{1}{q} < 1. \)

\[ d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}) \]

\[ = G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) \]

\[ \geq q \max \left\{ G(y_{3n+2}, y_{3n+3}, y_{3n+1}), G(y_{3n+2}, y_{3n+1}, y_{3n}) \right\} \]

\[ = q \max \{ d_{3n+1}, d_{3n}, d_{3n+1}, d_{3n} \} . \]

Thus we have \( d_{3n} \geq q d_{3n+1} \) so that \( d_{3n+1} \leq k d_{3n} . \)

\[ d_{3n+1} = G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \]

\[ = G(Sx_{3n+1}, Tx_{3n+2}, Rx_{3n+3}) \]

\[ \geq q \max \left\{ G(y_{3n+2}, y_{3n+3}, y_{3n+4}), G(y_{3n+2}, y_{3n+1}, y_{3n+3}) \right\} \]

\[ = q \max \{ d_{3n+2}, d_{3n+1}, d_{3n+1}, d_{3n+2} \} . \]

Thus we have \( d_{3n+1} \geq q d_{3n+2} \) so that \( d_{3n+2} \leq k d_{3n+1} . \)

Hence \( G( y_n, y_{n+1}, y_{n+2} ) \leq k G( y_n, y_{n+1}, y_{n+2} ) \)

\[ \leq k^2 G( y_{n-1}, y_n, y_{n+1} ) \]

\[ \vdots \]

\[ \leq k^n G(y_0, y_1, y_2) . \]

From \( (G_3) \), we have

\[ G( y_n, y_{n+1}, y_{n+1} ) \leq G( y_n, y_{n+1}, y_{n+2} ) \leq k^n G(y_0, y_1, y_2) . \]

From \( (G_5) \) for \( m > n \) we have

\[ G( y_n, y_{n}, y_{m} ) \leq G( y_n, y_{n+1}, y_{n+1} ) + G( y_{n+1}, y_{n+1}, y_{n+2} ) + \ldots + G( y_{m-1}, y_{m-1}, y_{m} ) \]

\[ \leq (k^n + k^{n+1} + \ldots + k^{m-1}) G(y_0, y_1, y_2) . \]
\[ \leq \frac{k^n}{1-\alpha} G(y_0, y_1, y_2) \]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty, \ m \rightarrow \infty. \]

Hence \{y_n\} is G-Cauchy.

Suppose f(X) is a G-complete subspace of X. Then there exist p, t \in X such that \(y_{3n+1} \rightarrow p = f(t)\).

Since \{y_n\} is G-Cauchy, it follows that \(y_{3n} \rightarrow p\) and \(y_{3n+2} \rightarrow p\).

\[ G(St, y_{3n+2}, y_{3n+3}) = G(St, Tx_{3n+2}, Rx_{3n+3}) \]
\[ \geq q \max \left\{ G(p, y_{3n+3}, y_{3n+4}), G(p, St, y_{3n+3}) \right\} \]
\[ \quad \geq q \max \left\{ G(Sp, y_{3n+3}, y_{3n+4}), G(Sp, Sp, y_{3n+3}) \right\}. \]

Letting \(n \rightarrow \infty\), we get
\[ G(St, p, p) \geq G(p, St, p). \]

Hence \(St = p\). Thus \(f(t) = St = p\).

Since \((f, S)\) is a weakly compatible pair, we have \(f(p) = Sp\).

\[ G(Sp, y_{3n+2}, y_{3n+3}) = G(Sp, Tx_{3n+2}, Rx_{3n+3}) \]
\[ \geq q \max \left\{ G(Sp, y_{3n+3}, y_{3n+4}), G(Sp, Sp, y_{3n+3}) \right\}. \]

Letting \(n \rightarrow \infty\), we get
\[ G(Sp, p, p) \geq q \max \{ G(Sp, p, p), G(Sp, Sp, p), G(p, p, Sp), 0\} \]
\[ \geq q \max \left\{ G(Sp, p, p), \frac{1}{2} G(Sp, p, p), 0 \right\}, \text{ since } G(p, p, Sp) \leq 2G(Sp, Sp, p) \]
\[ = q G(Sp, p, p). \]

Hence \(Sp = p\). Thus \(f(p) = Sp = p\). 

\[ \ldots \ldots (1) \]

Since \(p = Sp \in T(X)\), there exists \(v \in X\) such that \(p = Tv\).

\[ G(Sp, Tv, y_{3n+3}) = G(Sp, Tv, Rx_{3n+3}) \]
\[ \geq q \max \{ G(p, gv, y_{3n+4}), G(p, p, y_{3n+3}), G(gv, p, p), G(y_{3n+4}, y_{3n+3}, p) \}. \]

Letting \(n \rightarrow \infty\) we get, \(0 \geq q \max \{ G(p, gv, p), 0, G(gv, p, p), 0 \}\).

Hence \(G(p, gv, p) = 0\) so that \(gv = p\). Thus \(gv = Tv = p\).

Since \((g, T)\) is a weakly compatible pair, we have \(g(p) = Tp\).
\[ G(p, Tp, y_{3n+3}) = G(Sp, Tp, Rx_{3n+3}) \]
\[ \geq q \max \left\{ G(p, Tp, y_{3n+4}), G(p, p, y_{3n+3}), \right\} \]
\[ \geq q \max \left\{ G(Tp, Tp, p), G(y_{3n+4}, y_{3n+3}, Tp) \right\}. \]

Letting \( n \rightarrow \infty \) we get
\[ G(p, Tp, p) \geq q \max \{ G(p, Tp, p), 0, G(Tp, Tp, p), G(p, p, Tp) \} \]
\[ \geq q \max \left\{ G(p, Tp, p), \frac{1}{2} G(p, p, Tp) \right\}, \text{since } G(p, p, Tp) \leq 2 G(Tp, Tp, p) \]
\[ = q G(p, p, Tp). \]
Hence \( Tp = p \). Thus \( g p = Tp = p. \) \( \ldots (2) \)

Since \( p = gp \in R(X) \), there exists \( w \in X \) such that \( p = hw \).
\[ G(p, p, Rw) = G(Sp, Tp, Rw) \]
\[ \geq q \max \left\{ G(p, p, p), G(p, p, Rw), G(p, p, p), G(p, Rw, p) \right\} \]
\[ = q G(p, p, Rw). \]
Hence \( Rw = p \). Thus \( hw = Rw = p. \)

Since \( (h, R) \) is a weakly compatible pair, we have \( Rp = hp \).
\[ G(p, p, Rp) = G(Sp, Tp, Rp) \]
\[ \geq q \max \left\{ G(p, p, Rp), G(p, p, Rp), G(p, p, p), G(Rp, Rp, p) \right\} \]
\[ \geq q \max \left\{ G(p, p, Rp), \frac{1}{2} G(p, p, Rp) \right\}, \text{since } G(p, p, Rp) \leq 2 G(Rp, Rp, p) \]
\[ = q G(p, p, Rp). \]
Hence \( Rp = p \). Thus \( hp = Rp = p. \) \( \ldots (3) \)

From (1), (2) and (3) it follows that \( p \) is a common fixed point of \( S, T, R, f, g \) and \( h \).

Suppose \( p' \) is another common fixed point of \( S, T, R, f, g \) and \( h \).
\[ G(p, p, p') = G(Sp, Tp, Rp') \]
\[ \geq q \max \left\{ G(p, p, p'), G(p, p, p'), G(p, p, p), G(p', p', p) \right\} \]
\[ \geq q \max \left\{ G(p, p, p'), \frac{1}{2} G(p, p, p') \right\}, \text{since } G(p, p, p') \leq 2 G(p', p', p) \]
\[ = q G(p, p, p'). \]

\[ \text{pp. 118} \]
Hence \( p' = p \).

Thus \( p \) is a unique common fixed point of \( S, T, R, f, g \) and \( h \).

Similarly, the theorem holds if \( g(X) \) or \( h(X) \) is a \( G \)-complete subspace of \( X \).

Finally, we prove the following in the similar lines.

**THEOREM 2.2:** Let \((X, G)\) be a complete \( G \)-metric space and \( S, T, R, f, g, h : X \rightarrow X \) be mappings such that

\[
G(Sx, Ty, Rz) \geq q \min \left\{ G(fx, gy, hz), G(fx, Sx, Rz), \frac{G(fx, Sx, Ty)}{2}, \frac{G(fx, hz, Ty)}{2}, \frac{G(fx, Ty, Sx)}{2}, \frac{G(gy, Ty, Sx)}{2}, \frac{G(gy, hz, Sx)}{2}, \frac{G(gy, Sx, hz)}{2}, \frac{G(hz, Sx, Ty)}{2}, \frac{G(hz, Ty, Sx)}{2} \right\}
\]

or

\[
G(Sx, Ty, Rz) \geq q \cdot G(fx, gy, hz)
\]

for all \( x, y, z \in X \) and \( q > 1 \),

(2.2.2) \( h(X) \subseteq S(X) \), \( f(X) \subseteq T(X) \), \( g(X) \subseteq R(X) \),

(2.2.3) one of \( f(X) \), \( g(X) \) and \( h(X) \) is a \( G \)-complete subspace of \( X \),

(2.2.4) the pairs \((f, S)\), \((g, T)\) and \((h, R)\) are weakly compatible.

Then (a) one of the pairs \((f, S)\), \((g, T)\) and \((h, R)\) has a coincidence point in \( X \) or

(b) \( S, T, R, f, g \) and \( h \) have a unique common fixed point in \( X \).

The following example illustrates the Theorem 2.2.

**Example 2.3 :** Let \( X = [0, \infty) \) and \( G(x, y, z) = |x - y| + |y - z| + |z - x| \), \( \forall x, y, z \in X \).

Let \( S, T, R, f, g, h : X \rightarrow X \) be defined by \( Sx = \frac{x}{2} \), \( Tx = \frac{x}{4} \), \( Rx = x \),

\[
fx = \frac{x}{16}, \quad gx = \frac{x}{32}, \quad hx = \frac{x}{8}.
\]

Clearly (2.2.2) – (2.2.4) are satisfied. Also \( G(Sx, Ty, Rz) = 8 \cdot G(fx, gy, hz) \) for all \( x, y, z \in X \). Clearly “0” is the unique common fixed point of \( S, T, R, f, g \) and \( h \).

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