ON SOME REFINEMENTS OF BERNSTEIN TYPE INEQUALITIES

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Received 06 August, 2012; Revised 24 March, 2013

Abstract:
In this paper, we generalize some inequalities concerning to the Bernstein’s inequality for polynomials.

Mathematics Subject Classification(2000): 30C10, 30C15.

Keywords and Phrases: Polynomials, Inequalities, Zeros

INTRODUCTION
If $P(z)$ is a polynomial of degree $n$, then concerning the estimate of the maximum of $|P(z)|$ on the unit circle $|z|=1$ and the estimate of the maximum of $|P(z)|$ on a large circle $|z|=R>1$, we have

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \quad (1)$$
$$\max_{|z|=R>1}|P'(z)| \leq R^n \max_{|z|=1}|P(z)| \quad (2)$$

Inequality (1) is an immediate consequence of Bernstein’s theorem on the derivative of a trigonometric polynomial (for reference see [4]). Inequality (2) is a simple deduction from maximum modulus principle (see [3, p.346] or [2, Vol. i, p.137]).

In both (1) and (2) equality holds only for $P(z) = \alpha z^n$, $|\alpha| \neq 0$ that is, if and only if $P(z)$ has all its zeros at the origin. Recently it was shown by Frappier, Rahman and Ruscheweyh [1, theorem 8] that, If $P(z)$ is a polynomial of degree $n$, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{1 \leq k \leq 2n} \left| p(e^{ik\pi/n}) \right| .$$

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the example $P(z) = z^n + ia$, $a > 0$.

Now we have, If $P(z)$ is a polynomial of degree $n$, then for all real $\lambda$, and $R > 1$,

$$\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n-1}{2} [M_{\lambda} + M_{\lambda+\pi}] \quad (4)$$

where
\[ M_\lambda = \max_{1 \leq k \leq n} |P(e^{\frac{i(2k\pi + \lambda)}{n}})| \]

and \( M_{\lambda+\pi} \) is obtained by replacing \( \lambda \) by \( \lambda + \pi \) in \( M_\lambda \). The result is best possible and equality holds for \( P(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1 \).

**Theorem A:** If \( P(z) \) is a polynomial of degree \( n \), then for all real \( \lambda \), and \( R > 1 \),
\[
\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left[ M_\lambda + M_{\lambda+\pi} \right], \tag{5}
\]
where
\[
M_\lambda = \max_{1 \leq k \leq n} |P(e^{\frac{i(2k\pi + \lambda)}{n}})|
\]
and \( M_{\lambda+\pi} \) is obtained by replacing \( \lambda \) by \( \lambda + \pi \) in \( M_\lambda \).

**Theorem B:** If \( P(z) \) is a polynomial of degree \( n \), having all zeros in \( |z| \geq 1 \), then for all real \( \lambda \) and \( R > 1 \),
\[
\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left[ M_\lambda^2 + M_{\lambda+\pi}^2 \right]^{1/2}. \tag{6}
\]

**Theorem C:** If \( P(z) \) is a polynomial of degree \( n \) such that \( P(1)=0 \), then for \( 0 \leq \omega \leq n \)
\[
|P \left( 1 - \frac{\omega}{n} \right)| \leq \left| \left( 1 - \frac{\omega}{n} \right)^n \left[ P \left( \frac{1}{r} \right) \right] - \frac{1}{2} \{ M_0 + M_{\lambda+\pi} \} \right| + \frac{1}{2} \{ M_0 + M_{\lambda+\pi} \}. \tag{7}
\]

**MAIN RESULTS**

**Theorem 1:** If \( P(z) \) is a polynomial of degree \( n \), then for all real \( \lambda \) and \( R > r \geq 1 \)
\[
\max_{|z|=1} |P(rz) - P(z)| \leq \frac{R^n - r^n}{2} \left[ M_\lambda + M_{\lambda+\pi} \right] \tag{8}
\]

**Remark 1:** For \( r = 1 \), we get (5).

On dividing both sides of (8) by \( (R-r) \) and letting \( R \to r \), we get

**Corollary 1:** If \( P(z) \) is a polynomial of degree \( n \), then for all real \( \lambda \) and \( r \geq 1 \)
\[
\max_{|z|=1} |P'(rz)| \leq \frac{nr^{n-1}}{2} \left[ M_\lambda + M_{\lambda+\pi} \right]
\]

**Theorem 2:** If \( P(z) \) is a polynomial of degree \( n \), having all zeros in \( |z| \geq 1 \), then for all real \( \lambda \) and \( R > r \geq 1 \),
\[
\max_{|z|=1} |P(Rz) - P(rz)| \leq \frac{R^n - r^n}{2} \left[ M_\lambda^2 + M_{\lambda+\pi}^2 \right]^{1/2} \tag{9}
\]
Remark 2: For \( r = 1 \), we get (6).

Now dividing on both sides by \( (R - r) \) of (9) and letting \( R \to r \), we obtain

**Corollary 2:** If \( P(z) \) is a polynomial of degree \( n \), having all zeros in \( |z| \geq 1 \), then for all real \( \lambda \) and \( r \geq 1 \),

\[
\max_{|z|=1} |P'(rz)| \leq \frac{n r^{n-1}}{2} \left[ \frac{M_\lambda^2 + M_{\lambda+\pi}^2}{2} \right]^{1/2}
\]

**Theorem 3:** If \( P(z) \) is a polynomial of degree \( n \) such that \( P(1) = 0 \), then for \( 0 \leq \omega \leq n \) and \( r \geq 1 \)

\[
|P \left( 1 - \frac{\omega}{n} \right) | \leq \left( 1 - \frac{\omega}{n} \right)^n \left| P \left( \frac{1}{r} \right) \right| \left( \frac{1}{2} \left( M_0 + M_{\lambda+\pi} \right) \right) + \frac{1}{2} \left( M_0 + M_{\lambda+\pi} \right)
\]

**Remark 3:** For \( r = 1 \), we get (7).

**LEMMA**
To prove these results, we use the following lemmas:

**Lemma 1:** If \( P(z) \) is a polynomial of degree \( n \), then for all real \( \lambda \),

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left[ M_\lambda + M_{\lambda+\pi} \right],
\]

where

\[
M_\lambda = \max_{|z|=1} \left| P(e^{i(2k\pi + \lambda)}) \right|
\]

and \( M_{\lambda+\pi} \) is obtained by replacing \( \lambda \) by \( \lambda + \pi \) in \( M_\lambda \).

**Lemma 2:** If \( P(z) \) is a polynomial of degree \( n \), having all zeros in \( |z| \geq 1 \), then for all real \( \lambda \)

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left[ M_\lambda^2 + M_{\lambda+\pi}^2 \right]^{1/2}.
\]

The result is sharp and equality holds for

\[
P(z) = z^n + e^{i\alpha}.
\]

**PROOF OF THEOREMS**

**Proof of Theorem 1:** Applying (2) to the polynomial \( P'(z) \), which is of degree \( n - 1 \), we get

\[
\left| p'(se^{i\theta}) \right| \leq s^{n-1} \max_{|z|=1} |P'(z)|
\]

Therefore by Lemma 1, we have

\[
\max_{|z|=1} |P'(sz)| \leq \frac{n s^{n-1}}{2} \left[ M_\lambda + M_{\lambda+\pi} \right]
\]

Hence for each \( \theta, 0 \leq \theta < 2\pi \) and \( R > r \geq 1 \), we have

\[
\left| P(Re^{i\theta}) - P(re^{i\theta}) \right| = \left|e^{i\theta} P'(se^{i\theta}) ds \right|
\]
\[ |P(Rz) - P(rz)| \leq \frac{n}{2} \left[ M_\lambda + M_{\lambda+\pi} \right] \int_r^R s^{n-1} ds \]
\[ |P(Rz) - P(rz)| \leq \frac{R^n - r^n}{2} \left[ M_\lambda + M_{\lambda+\pi} \right], \]
where
\[ M_\lambda = \max_{1 \leq k \leq n} \left| P(e^{i(2k\pi + \lambda/n)}) \right| \]
and \( M_{\lambda+\pi} \) is obtained by replacing \( \lambda \) by \( \lambda + \pi \) in \( M_\lambda \).

**Proof of Theorem 2:** Applying (2) to the polynomial \( P'(z) \) of degree \( n-1 \), we get
\[ |p'(se^{i\vartheta})| \leq s^{-1} \max_{|s|=1} |p'(z)| \]
Using lemma 2, we have
\[ \max_{|z|=1} |P'(se^{i\vartheta})| \leq s^{n-1} \left( \frac{n}{2} \right)^1 \left( M_\lambda^2 + M_{\lambda+\pi}^2 \right)^{1/2} \]
Hence for each \( \vartheta, 0 \leq \vartheta < 2\pi \) and \( R > r \geq 1 \), we have, for \( R > r \geq 1 \)
\[ |P(Re^{i\vartheta}) - P(re^{i\vartheta})| = \left| \int_r^R e^{i\vartheta} P'(se^{i\vartheta}) ds \right| \]
This implies
\[ \max_{|z|=1} |P(Rz) - P(rz)| \leq \frac{R^n - r^n}{2} \left[ M_\lambda^2 + M_{\lambda+\pi}^2 \right]^{1/2}, \]
where
\[ M_\lambda = \max_{1 \leq k \leq n} \left| P(e^{i(2k\pi + \lambda/n)}) \right| \]
and \( M_{\lambda+\pi} \) is obtained by replacing \( \lambda \) by \( \lambda + \pi \) in \( M_\lambda \).

**Proof of Theorem 3:** If \( Q(z) = z^n P(z) \), then \( |Q(z)| = |P(z)| \) for \( |z| = 1 \) and by hypothesis, we have \( |Q(1)| = |P(1)| = 0 \). Applying theorem 1 to \( Q(z) \) with \( \lambda = 0 \), we get for \( R > r \geq 1 \)
\[ |Q(Rz) - Q(rz)| \leq \frac{R^n - r^n}{2} \left[ M_0 + M_\pi \right] \]
This implies for \( R > r \geq 1 \)
\[ |P(\frac{r}{R})| \leq \frac{1}{2R^n} (R^n - r^n) \left[ M_0 + M_\pi \right] + \frac{n}{R^n} |P(\frac{r}{R})| \]
If \( 0 < \omega \leq n \) then \( \left( 1 - \frac{\omega}{n} \right)^{-1} > 1 \) and therefore and in particular, we have
\[ |P \left( 1 - \frac{\omega}{n} \right)| \leq \left[ \left( 1 - \frac{\omega}{n} \right) R \right]^n \left| P \left( \frac{1}{R} \right) \right| - \frac{1}{2} \left[ M_0 + M_{\lambda+\pi} \right] \] + \frac{1}{2} \left[ M_0 + M_{\lambda+\pi} \right]. \]
Hence the result.

**ACKNOWLEDGEMENT**
Authors are highly thankful to the referees for their valuable suggestions/comments.
REFERENCES