ON THE INVERSE LAPLACE TRANSFORM

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ABSTRACT

In this work we show that the Tuan-Duc formula used to invert the Laplace transform is equivalent to the expression given by Post-Widder.

Keywords: Laplace transform; Bromwich formula; Post-Widder and Tuan-Duc equations

INTRODUCTION

The Laplace transform of \( f(x) \) is defined as [1]:

\[
L\left[ f(x) \right] = F(s) = \int_0^\infty e^{-sx} f(x) \, dx,
\]

and the inverse problem is to determine \( f(t) \) for a given \( F(s) \). Bromwich [2, 3] gave the expression:

\[
f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} F(s) \, ds,
\]

where the integration is performed in the complex plane along the straight line \( x = \sigma \).

But due to the interest to get the inverse Laplace transform without complex variable, Post [4] and Widder [5, 7] found the following formula in a real variable:

\[
f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left( \frac{n!}{t} \right)^{n+1} \left[ \frac{d^n F}{ds^n} \right]_{s=n/t}
\]

In Sec. 2 it is given an alternative form for (3) to obtain a procedure for the recent expression of Tuan-Duc [6]:

\[
f(t) = \lim_{n \to \infty} \prod_{k=1}^n \left( 1 + \frac{t}{k} \frac{d}{dt} \right) \left[ \frac{n!}{t} F\left( \frac{n}{t} \right) \right]
\]

and there it will be clear that (4) is another manner to write (3).

THE POST-WIDDER AND TUAN-DUC FORMULAE

First, we know that [1]:

\[
L\left[ x^n \right] = \int_0^\infty e^{-sx} x^n \, dx = \frac{n!}{s^{n+1}},
\]
where in $s = \frac{n}{t}$ implies:

$$
\frac{n^{n+1}}{n! t^{n+1}} \int_0^n e^{-\frac{n^x}{x}} x^n dx = 1
$$

(6)

On the other hand, the binomial theorem of Newton gives:

$$(x-t)^r = \sum_{k=0}^r \binom{r}{k} x^k (-t)^{r-k},$$

(7)

such that for $r \geq 1$:

$$
\frac{n^{n+1}}{n! t^{n+1}} \int_0^n e^{-\frac{n^x}{x}} x^n (x-t)^r dx
$$

$$
= \frac{n^{n+1}}{n! t^{n+1}} \int_0^n e^{-\frac{n^x}{x}} x^n \sum_{k=0}^r \binom{r}{k} x^k (-t)^{r-k} dx,
$$

(8)

where we employ the gamma function

$$
\Gamma (n+k+1) = \int_0^n e^u u^{n+k} du = (n+k)!
$$

and our integral adopts the form:

$$
t^r \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} \frac{(n+1)(n+2) \ldots (n+k)}{n^k}
$$

$$
\rightarrow t^r \sum_{k=0}^r \binom{r}{k} (1)^k (-1)^{r-k} = t^r (1-1)^r = 0
$$

The expansion of $f(x)$ in Taylor series around $x = t$:

$$
f(x) = f(t) + \sum_{r=0}^\infty \frac{f^{(r)}(t)}{r!} (x-t)^r
$$

(9)

together with (6) and (8), leads to the relation:

$$
f(t) = \lim_{n \to \infty} n^{n+1} \frac{n^n}{n! t^{n+1}} \int_0^n e^{-\frac{n^x}{x}} x^n f(x) dx
$$

(10)

Further, from (1) it is immediate that:

$$
\left( t^2 \frac{d}{dt} \right) F \left( \frac{n}{t} \right) = n \int_0^n e^{-\frac{n^x}{x}} x^n f(x) dx,
$$
and after successive applications of the operator \( \left( t^2 \frac{d}{dt} \right)_n \) we get the identity:

\[
\frac{1}{(n-1)!} t^{n+1} \left( t^2 \frac{d}{dt} \right)^n F \left( \frac{n}{t} \right) = \frac{n^{n+1}}{n!} \int_0^\infty e^{-\frac{x^n}{t}} x^n f(x) \, dx,
\]

which by substitution into (10) gives the inversion formula:

\[
f(t) = \lim_{n \to \infty} \frac{1}{n-1} \frac{1}{t^{n+1} \left( t^2 \frac{d}{dt} \right)^n F \left( \frac{n}{t} \right)},
\]

allowing the construction of \( f(t) \) from \( F(s) \).

It is simple to prove that (12) leads to the expression (3) deduced by Post [4]–Widder [5, 7]; in fact, with \( s = \frac{n}{t} \):

\[
t^2 \frac{d}{dt} F \left( \frac{n}{t} \right) = t^2 \frac{d}{dt} \left[ \frac{dF}{ds} \right]_{s = \frac{n}{t}} = -n \left[ \frac{dF}{ds} \right]_{s = \frac{n}{t}},
\]

so

\[
\left( t^2 \frac{d}{dt} \right)^n F \left( \frac{n}{t} \right) = (-1)^n n^n \left[ \frac{d^n F}{ds^n} \right]_{s = \frac{n}{t}},
\]

which in (12) implies (3). That is, the Post-Widder relation is the simplified form of (12).

From (1), it can be shown that:

\[
\left( 1 + t \frac{d}{dt} \right) \left[ \frac{n}{t} F \left( \frac{n}{t} \right) \right] = \frac{n^3}{t^2} \int_0^\infty e^{-\frac{x}{t}} x \, f(x) \, dx,
\]

then

\[
\left( 1 + \frac{t}{2} \frac{d}{dt} \right) \left[ 1 + t \frac{d}{dt} \right] \left[ \frac{n}{t} F \left( \frac{n}{t} \right) \right] = \frac{n^3}{2t^2} \int_0^\infty e^{-\frac{x}{t}} x^2 \, f(x) \, dx, \text{ etc.}
\]

and the identity:

\[
\prod_{k=1}^n \left( 1 + \frac{t}{k} \frac{d}{dt} \right) \left[ \frac{n}{t} F \left( \frac{n}{t} \right) \right] = \frac{n^{n+1}}{n!} \int_0^\infty e^{-\frac{x^n}{t}} x^n f(x) \, dx,
\]

is generated, which in (10) gives the Tuan-Duc formula [6] as shown in (4).

Therefore, (3), (4) and (12) are equivalent among them because:

\[
\frac{(-1)^n}{n!} \left( \frac{n}{t} \right)^{n+1} \left[ \frac{d^n F}{ds^n} \right]_{s = \frac{n}{t}} = \prod_{k=1}^n \left( 1 + \frac{t}{k} \frac{d}{dt} \right) \left[ \frac{n}{t} F \left( \frac{n}{t} \right) \right]
\]

\[
= \frac{1}{(n-1)!} \left( t^2 \frac{d}{dt} \right)^n F \left( \frac{n}{t} \right),
\]

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\]
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REFERENCES