FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS IN G-METRIC SPACE

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ABSTRACT
In this paper, we prove common fixed point theorems for a pair of occasionally weakly compatible maps in Symmetric G-metric space. Our results generalize and extend several relevant common fixed point theorems from the literature.

Key words: Symmetric G-metric space, occasionally weakly compatible maps, weakly compatible maps.

Subject classification: 2000 AMS: 47H10, 54H25

INTRODUCTION
In 1992, Dhage[1] introduced the concept of D–metric space. Recently, Mustafa and Sims[5] shown that most of the results concerning Dhage’s D–metric spaces are invalid. Therefore, they introduced G–metric space. For more details on G–metric spaces, one can refer to the papers [5]-[8].

In 2006, Mustafa and Sims[6] introduced the concept of G-metric spaces as follows:

Definition 1.1.[6] Let X be a nonempty set, and let \( G: X \times X \times X \to R^+ \) be a function satisfying the following axioms:

\( G(x, y, z) = 0 \) if \( x = y = z \),

\( 0 < G(x, x, y) \), for all \( x, y \in X \) with \( x \neq y \),

\( G(x, x, y) \leq G(x, y, z) \), for all \( x, y, z \in X \) with \( z \neq y \),

\( G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots \) (symmetry in all three variables) and

\( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \), (rectangle inequality)

then the function \( G \) is called a generalized metric, or, more specifically a G–metric on \( X \) and the pair \((X, G)\) is called a G – metric space.

If condition \((G6)\) also satisfied then \((X, G)\) is called Symmetric G-metric space.

Definition 1.2.[6] Let \((X, G)\) be a G–metric space, and let \( \{x_n\} \) a sequence of points in \( X \), a point ‘x’ in \( X \) is said to be the limit of the sequence \( \{x_n\} \) if \( \lim_{m,n\to \infty} G(x, x_m, x_n) = 0 \), and one says that sequence \( \{x_n\} \) is G–convergent to \( x \).

Thus, that if \( x_n \to x \) or \( \lim_{n\to \infty} x_n = x \) in a G-metric space \((X, G)\) then for each \( \varepsilon > 0 \), there exists a positive integer \( N \) such that \( G(x, x_m, x_n) < \varepsilon \) for all \( m, n \geq N \).
Proposition 1.1.[6] Let \((X, G)\) be a \(G\)–metric space. Then the following are equivalent:

1. \(\{x_n\}\) is \(G\)-convergent to \(x\),
2. \(G(x_n, x_m, x) \to 0\) as \(n \to \infty\),
3. \(G(x_n, x, x) \to 0\) as \(n \to \infty\),
4. \(G(x_m, x_n, x) \to 0\) as \(m, n \to \infty\).

Definition 1.3.[6] Let \((X, G)\) be a \(G\)–metric space. A sequence \(\{x_n\}\) is called \(G\)–Cauchy if, for each \(\varepsilon > 0\) there exists a positive integer \(N\) such that \(G(x_n, x_m, x_l) < \varepsilon\) for all \(n, m, l \geq N\); i.e. if \(G(x_n, x_m, x_l) \to 0\) as \(n, m, l \to \infty\).

Proposition 1.2.[6] If \((X, G)\) is a \(G\)–metric space then the following are equivalent:

1. The sequence \(\{x_n\}\) is \(G\)-Cauchy,
2. for each \(\varepsilon > 0\), there exist a positive integer \(N\) such that \(G(x_n, x_m, x_l) < \varepsilon\) for all \(n, m \geq N\).

Proposition 1.3.[6] Let \((X, G)\) be a \(G\)–metric space. Then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

Definition 1.4.[6] A \(G\)–metric space \((X, G)\) is said to be \(G\)–complete if every \(G\)-Cauchy sequence in \((X, G)\) is \(G\)-convergent in \(X\).

Proposition 1.4.[6] A \(G\)–metric space \((X, G)\) is \(G\)–complete if and only if \((X, d_G)\) is a complete metric space.

Proposition 1.5.[6] Let \((X, G)\) be a \(G\)–metric space. Then, for any \(x, y, z, a\) in \(X\) it follows that:

1. If \(G(x, y, z) = 0\), then \(x = y = z\),
2. \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\),
3. \(G(x, y, z) \leq 2G(y, x, x)\),
4. \(G(x, y, z) \leq G(x, a, a) + G(a, a, z)\),
5. \(G(x, y, z) \leq 2G(x, a, a) + G(x, a, z) + G(a, y, z)\),
6. \(G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))\).

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

Definition 1.5.[2] A pair of self mappings \((f, g)\) of a metric space is said to be weakly compatible if they commute at the coincidence points i.e. \(T u = S u\) for some \(u\) in \(X\), then \(T S u = S T u\).

Definition 1.6. Let \((X, G)\) be a Symmetric \(G\)-metric space. \(f\) and \(g\) be self maps on \(X\). A point \(x\) in \(X\) is called a coincidence point of \(f\) and \(g\) iff \(f x = g x\). In this case, \(w = f x = g x\) is called a point of coincidence of \(f\) and \(g\).

Definition 1.7.[3] A pair of self mappings \((f, g)\) of a Symmetric \(G\)-metric space \((X, G)\) is said to be weakly compatible if they commute at the coincidence points i.e., if \(fu = gu\) for some \(u\) in \(X\), then \(fg u = gf u\).

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 1.8.[3] Two self mappings \(f\) and \(g\) of a Symmetric \(G\)-metric space \((X, G)\) are
said to be occasionally weakly compatible (owc) iff there is a point \( x \) in \( X \) which is coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute.

**Lemma 1.1[3]:** Let \( (X, G) \) be a Symmetric \( G \)-metric space. \( f \) and \( g \) be self maps on \( X \) and let \( f \) and \( g \) have a unique point of coincidence, \( w = fx = gx \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

**MAIN RESULTS**

Following to Matkowski[5], let \( \Phi \) be the set of all functions \( \phi \) such that
\[
\phi : [0, +\infty) \rightarrow [0, +\infty)
\]
be a non-decreasing function with \( \lim_{t \to +\infty} \phi^n(t) = 0 \) for all \( t \in [0, +\infty) \).

If \( \phi \in \Phi \), then \( \phi \) is called \( \Phi \)-map. If \( \phi \) is \( \Phi \)-map, then it is an easy matter to show that
\[(A) \quad \phi(t) < t \quad \text{for all} \quad t \in [0, +\infty) ;
(B) \quad \phi(0) = 0.
\]

From now unless otherwise stated, we mean by \( \phi \) the \( \Phi \)-map. Now, we introduce and prove our result.

**Theorem 2.1:** Let \( (X, G) \) be a Symmetric \( G \)-metric space. If \( f \) and \( g \) are owc self maps on \( X \) and
\[G(fx, fy, fy) \leq \phi \left[ \max \{ G(gx, gy, gy), G(gx, fy, fy), G(gy, fx, fx), G(gy, fy, fy) \} \right] \quad (2.1)\]
for all \( x, y \in X \). Then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Since \( f \) and \( g \) are owc, there exist a point \( u \in X \) such that \( fu = gu \) and \( gfu = gfu \).

We claim that \( fu \) is the unique common fixed point of \( f \) and \( g \). We first assert that \( fu \) is a fixed point of \( f \).

For, if \( ffu \neq fu \), then from equation (2.1), we get
\[G(fu, ffu, ffu) \leq \phi \left[ \max \{ G(gu, gfu, gfu), G(gu, ffu, ffu), G(gfu, fu, fu), G(gfu, ffu, ffu) \} \right]
= \phi \left[ \max \{ G(fu, ffu, ffu), G(ffu, fu, fu), G(ffu, fu, fu), G(ffu, ffu, ffu) \} \right]
= \phi \left[ \max \{ G(fu, ffu, ffu), G(ffu, ffu, fu), G(ffu, fu, fu), 0 \} \right]
= \phi \left[ \max \{ G(fu, ffu, ffu), G(ffu, fu, ffu), G(ffu, fu, fu) \} \right]
= \phi \left[ G(ffu) \right] < \phi \left[ G(fu) \right]
\]
a contradiction. So \( ffu = fu \) and \( ffu = gfu = gfu = fu \). Hence \( fu \) is a common fixed point of \( f \) and \( g \).

Now we prove uniqueness. Suppose that \( u, v \in X \) such that \( fu = gu = u \) and \( fv = gv = v \) and \( u \neq v \). Then from equation (2.1),
\[G(u, v, v) = G(fu, fv, fv) \leq \phi \left[ \max \{ G(gu, gv, gv), G(gu, fv, fv), G(gv, fu, fu), G(gv, fv, fv) \} \right]
= \phi \left[ \max \{ G(u, v, v), G(u, v, v), G(v, u, u), G(v, v, v) \} \right]
= \phi \left[ \max \{ G(u, v, v), G(u, v, v), 0 \} \right]
= \phi \left[ \max \{ G(u, v, v), G(u, v, v), 0 \} \right]
= \phi \left[ G(u, v, v) \right] < \phi \left[ G(u, v, v) \right]
\]
a contradiction. So \( u = v \). Therefore, the common fixed point of \( f \) and \( g \) is unique.

**Theorem 2.2:** Let \( (X, G) \) be a Symmetric \( G \)-metric space. Suppose that \( f, g, S, T \) are self maps on \( X \) and that the pairs \{ \( f, S \) \} and \{ \( g, T \) \} are each owc. If
\[G(fx, gy, gy) \leq \max \{ G(Sx, Ty, Ty), G(Sx, fx, fx), G(Ty, gy, gy), G(Sx, gy, gy), G(Ty, fx, fx) \}, \quad (2.2)\]
for all \( x, y \in X \). Then \( f, g, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** By hypothesis, there exists points \( x, y \in X \) such that \( fx = Sx \) and \( gy = Ty \). We claim that \( fx = gy \). For, otherwise, by (2.2)

\[
G(fx, gy, gy) < \max \{ G(Sx, Ty), G(Sx, fx, fx), G(Ty, gy, gy), G(Sx, gy, gy), G(Ty, fx, fx) \} \\
= \max \{ G(fx, gy, gy), G(fx, fx, fx), G(gy, gy, gy), G(fx, gy, gy), G(gy, fx, fx) \} \\
= \max \{ G(fx, gy, gy), 0, 0, G(fx, gy, gy), G(gy, fx, fx) \} \\
= G(fx, gy, gy)
\]

a contradiction. This implies that \( fx = gy \). So \( fx = Sx = gy = Ty \). Moreover, if there is another point \( z \) such that \( fz = Sz \), then, using (2.2) it follows that \( fz = Sz = gy = Ty \) or \( fx = fz \) and \( w = fx = Sx \) is the unique point of coincidence of \( f \) and \( S \). Then by Lemma 1.1, it follows that \( w \) is the unique common fixed point of \( f \) and \( S \). By symmetry, there is a unique common fixed point \( z \in X \) such that \( z = gz = Tz \).

Now, we claim that \( w = z \). Suppose that \( w \neq z \). Using (2.2),

\[
G(w, z, z) = G(fw, gz, gz) \\
< \max \{ G(Sw, Tz, Tz), G(Sw, fw, fw), G(Tz, gz, gz), G(Sw, gz, gz), G(Tz, fw, fw) \} \\
G(w, z, z) < \max \{ G(w, z, z), G(w, w, w), G(z, z, z), G(w, z, z), G(z, w, w) \} \\
= \max \{ G(w, z, z), 0, 0, G(w, z, z), G(z, z, w) \} \\
= \max \{ G(w, z, z), G(w, z, z), G(w, z, z), G(w, z, z) \} = G(w, z, z)
\]

This is a contradiction. Therefore \( w = z \) and \( w \) is a unique point of coincidence of \( f, g, S \) and \( T \). By Lemma 1.1, \( w \) is the unique common fixed point of \( f, g, S \) and \( T \).

**Corollary 2.1:** Let \( (X, G) \) be a Symmetric \( G \)-metric space. Suppose that \( f, g, S \) and \( T \) are self maps on \( X \) and that the pairs \( \{ f, S \} \) and \( \{ g, T \} \) are each owc. If \( G(fx, gy, gy) \leq h m(x, y, y) \) where

\[
m(x, y, y) = \max \{ G(Sx, Ty, Ty), G(Sx, fx, fx), G(Ty, gy, gy), G(Sx, gy, gy), G(Ty, fx, fx) \}/2,
\]

for all \( x, y \in X \) and \( 0 \leq h < 1 \), then \( f, g, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** Since (2.3) is a special case of (2.2), the result follows immediately from Theorem 2.2.

**Theorem 2.3.** Let \( A, B, S \) and \( T \) be self maps of Symmetric \( G \)-metric space \( (X, G) \), satisfying the following conditions:

(2.4) \( A(X) \subset T(X) \), \( B(X) \subset S(X) \),
(2.5) pairs \( (A, S) \) or \( (B, T) \) satisfies property \( E.A. \),
(2.6) for all \( x, y \in X \),

\[
G(Ax, By, By) < \phi \left[ \max \{ G(Sx, Ty, Ty), G(Sx, By, By), G(Ty, By, By) \} \right]
\]

where \( \phi \in \Phi \). If one of \( A(X), B(X), S(X) \) or \( T(X) \) is complete subsets of \( X \) then pairs \( (A, S) \) and \( (B, T) \) have coincidence point.

Further, if \( (A, S) \) and \( (B, T) \) are weakly compatible then \( A, B, S \) and \( T \) have unique common fixed point in \( X \).

**Proof:** Suppose the pair \( (B, T) \) satisfies the property \( (E.A.) \). Then there exists a sequence \( \{ x_n \} \) in \( X \) such that

\[
\lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = p \text{ for some } p \in X.
\]

Since \( B(X) \subset S(X) \), there exists a sequence \( \{ y_n \} \) in \( X \) such that

\[
Bx_n = Sx_n = p. \text{ Hence } \lim_{n \to \infty} Sx_n = p.
\]

We shall show that \( \lim_{n \to \infty} Ay_n = p \).

From (2.6), we have

\[
G(Ay_n, Bx_n, Bx_n) < \phi \left[ \max \{ G(Sy_n, Tx_n, Tx_n), G(Sy_n, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n) \} \right]
\]
Taking limit as $n \to \infty$, we get
\[
\lim_{n \to \infty} G(Ay_n, p, p) < \phi \left( \max \{ G(p, p, p), G(p, p, p), G(p, p, p) \} \right)
\]
\[
= \phi \left( \max \{ 0, 0, 0 \} \right) = \phi (0) = 0.
\]
This implies, $\lim_{n \to \infty} Ay_n = p$.

Thus we have, $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = p$.

Suppose that $S(X)$ is a complete subspace of $X$. Then $p = Su$ for some $u \in X$.  

Subsequently, we have
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = p = Su
\]

Now, we shall show that $Au = Su$.

From (2.6), we have
\[
G(Au, Bx_n, Bx_n) < \phi \left( \max \{ G(Su, Tx_n, Tx_n), G(Su, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n) \} \right)
\]

Taking limit as $n \to \infty$ we get
\[
G(Au, Su, Su) < \phi \left( \max \{ G(p, p, p), G(p, p, p), G(p, p, p) \} \right)
\]
\[
= \phi \left( \max \{ 0, 0, 0 \} \right) = \phi (0) = 0.
\]

Thus, we have $Au = Su$. Therefore $(A, S)$ have coincidence point.

The weak compatibility of $A$ and $S$ implies that $ASu = SAu$ and thus $AAu = ASu = SAu = SSu$.

As $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$.

Suppose not, from (2.6), we have
\[
G(Au, Bv, Bv) < \phi \left( \max \{ G(Su, Tv, Tv), G(Su, Bv, Bv), G(Tv, Bv, Bv) \} \right)
\]
\[
= \phi \left( \max \{ 0, G(Au, Bv, Bv), G(Au, Bv, Bv) \} \right)
\]
\[
= \phi \left( G(Au, Bv, Bv) \right) < G(Au, Bv, Bv),
\]

this implies, $Au = Bv$.

Hence, $Tv = Bv$. Therefore $(B, T)$ have coincidence point.

Thus we have $Au = Su = Tv = Bv$.

The weak compatibility of $B$ and $T$ implies that $BTv = TBv = TTv = BBv$.

Finally, we show that $Au$ is the common fixed point of $A, B, S$ and $T$.

From (2.6), suppose $Au \neq AAu$, we have
\[
G(Au, AAu, AAu) = G(Au, Au, AAu) \quad \{ \text{by definition of symmetric space} \}
\]
\[
= G(AAu, Bv, Bv) < \phi \left( \max \{ G(SAu, Tv, Tv), G(SAu, Bv, Bv), G(Tv, Bv, Bv) \} \right)
\]
\[
= \phi \left( \max \{ G(AAu, Bv, Bv), G(AAu, Bv, Bv), G(Bv, Bv, Bv) \} \right)
\]
\[
= \phi \left( \max \{ G(AAu, Bv, Bv), G(AAu, Bv, Bv), 0 \} \right)
\]
\[
= \phi \left( G(AAu, Bv, Bv) \right) < G(AAu, Bv, Bv),
\]

This gives, $AAu = Bv = Au$ and thus $AAu = Au$.

Therefore, $Au = AAu = SAu$ is the common fixed point of $A$ and $S$.

Similarly, we prove that $Bv$ is the common fixed point of $B$ and $T$. Since $Au = Bv$, $Au$ is common fixed point of $A, B, S$ and $T$. The proof is similar when $T(X)$ is assumed to be a complete subspace of $X$. The cases in which $A(X)$ or $B(X)$ is a complete subspace of $X$ are similar to the cases in which $T(X)$ or $S(X)$, respectively is complete subspace of $X$ as $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Finally now we show that the common fixed point is unique. If possible, let $x_0$ and $y_0$ be two common fixed points of $A, B, S$ and $T$. Suppose $x_0 \neq y_0$, then by condition (2.6), we have
\[
G(x_0, y_0, y_0) = G(Ax_0, By_0, By_0)
\]
\[
< \phi \left( \max \{ G(Sx_0, Ty_0, Ty_0), G(Sx_0, By_0, By_0), G(Ty_0, By_0, By_0) \} \right),
\]
\[
\phi \left( \max \{ G(x_0, y_0, y_0), G(y_0, x_0, y_0), G(y_0, y_0, x_0) \} \right)
\]
\[
\phi \left( G(x_0, y_0, y_0) \right) < G(x_0, y_0, y_0),
\]
this implies \( x_0 = y_0 \).

Therefore, the mappings \( A, B, S \) and \( T \) have a unique common fixed point.

**Corollary 2.2.** Let \( A, B \) and \( S \) be self maps of Symmetric \( G \)-metric space \((X, G)\), satisfying the following conditions:

1. \( A(X) \subset S(X), B(X) \subset S(X) \),
2. \( G(Ax, By, By) < \phi \left( \max \{ G(Sx, Sy, Sy), G(Sx, By, By), G(Sy, By, By) \} \right) \)
   where \( \phi \in \Phi \).

If one of \( A(X) \), \( B(X) \) or \( S(X) \) is complete subsets of \( X \) then pairs \((A, S)\) and \((B, S)\) have coincidence point.

Further, if \((A, S)\) and \((B, S)\) are weakly compatible then \( A, B \) and \( S \) have unique common fixed point in \( X \).

**Proof:** Take \( T = S \) in Theorem 2.3.

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