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FIXED FUZZY POINT THEOREMS IN FUZZY METRIC SPACE

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ABSTRACT
In this paper, we have established some fixed fuzzy point theorems and common fixed fuzzy point theorems for fuzzy mappings satisfying a contractive type condition other than fuzzy Banach contractive type condition in complete fuzzy metric spaces.

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Key words: Fuzzy Set, Fuzzy mapping, Fixed fuzzy point, Fuzzy metric space.

INTRODUCTION
In many scientific and engineering applications, concept of fuzzy set plays an important role. In mathematical programming, problems are expressed as optimizing some goal functions under given certain constraints. There are some real-life problems having multiple objectives. Fuzzy sets are one of the possible methods to get feasible solutions that bring us to optimum of all objective functions. The concept of fuzzy set was introduced initially by Zadeh [10] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and its applications. Helipern first introduced the concept of fuzzy mappings and proved a fixed-point theorem for fuzzy mappings [3].

Since then, many fixed-point theorems for fuzzy mappings have been obtained by many authors [5, 1]. Kramosil and Michalek [4] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened an avenue for further development of analysis in such spaces. Later on, it is modified that a few concepts of mathematical analysis have been developed by George and Veeramani [2]. Many authors have introduced the concept of fixed point theorems in fuzzy metric space in different ways [7, 8].

In this paper, we have established some fixed fuzzy point theorems and common fixed fuzzy point theorems for fuzzy mappings satisfying a contractive type condition other than fuzzy Banach contractive type condition in complete fuzzy metric spaces.

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PRELIMINARIES

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 [9]: A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t- norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a \quad \forall\ a \in [0,1]$;

(iv) $a * b \le c * d$ whenever $a \le c,\ b \le d$ and $a, b, c, d \in [0,1]$;

Definition 2.2 [2]: The 3 – tuple $(X, \mu, *)$ is called a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t- norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying conditions:

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, s) * \mu(y, z, t) \le \mu(x, z, s + t)$;

(v) $\mu(x, y, .) : (0, \infty) \to (0, 1]$ is continuous; for all $x, y, z \in X$ and $t, s > 0$.

It is noted that $\mu(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Let X be a metric linear space and $(X, \mu, *)$ be a fuzzy metric space. A fuzzy set of X is an element of I^X where $I = [0,1]$. For $A, B \in I^X$ we denote $A \subseteq B$ if and only if $A(x) \le B(x)$ for each $x \in X$. For $\alpha \in (0,1]$ the fuzzy point x_α of X is the fuzzy set of X given by $x_\alpha(y) = \alpha$ if $y = x$ and $x_\alpha(y) = 0$ else [1]. The α - level set of A, denote by A_α, is defined by

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$$A_\alpha = \{ x \in X : A(x) \geq \alpha \} \text{ if } \alpha \in (0,1]$$

$$A_\alpha = \overline{\{ x \in X : A(x) \geq 0 \}}$$

where $\overline{B}$ denotes the closure of the (non-fuzzy) set B.

Heilpern [3] called a fuzzy mapping, a mapping from the set of X in to a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ if and only if A_α is a compact and convex in X for each $\alpha \in [0,1]$ and $\sup\{A(x) : x \in X\} = 1$. In this context, we have given the following definitions:

Definition 2.3: Let $A, B \in W(X)$ and $\alpha \in [0,1]$. Define

$$P_\alpha(A, B, t) = \sup\{ \mu(a, b, t) : a \in A_\alpha, b \in B_\alpha \text{ and } t > 0 \}$$

$$D_\alpha(A, B, t) = H(A_\alpha, B_\alpha, t)$$

$$H(A_\alpha, B_\alpha, t) = \inf\left\{ \inf_{x \in B_\alpha} \sup_{y \in A_\alpha} \mu(x, y, t), \inf_{x \in A_\alpha} \sup_{y \in B_\alpha} \mu(x, y, t) \right\}$$

The function P_α is called a α - space. It is easy to see that P_α is non-decreasing function of α. H is the Hausdorff fuzzy metric.

Notation 2.4: Let X be a metric space and $\alpha \in [0,1]$. Consider the following family $W_\alpha(X)$:

$$W_\alpha(X) = \{ A \in I^X : A_\alpha \text{ is nonempty, compact and convex} \}$$

Definition 2.5: Let x_α be a fuzzy point of X. We will say that x_α is a **fixed fuzzy point** of the fuzzy mapping F over X if $\{x_\alpha\} \subset F(x)$ (i.e., the fixed degree of x is the least α).

Definition 2.6: Let $(X, \mu, *)$ be a fuzzy metric space, $x \in X$, $r \in (0,1)$, $t > 0$,

$B(x, r, t) = \{ y \in X / \mu(x, y, t) > 1 - r \}$. Then $B(x, r, t)$ is called on **open ball** centered at x of radius r with respect to t.

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Definition 2.7: Let $(X, \mu, *)$ be a fuzzy metric space and $P \subseteq X$. P is said to be a **closed set** in $(X, \mu, *)$ if and only if sequence $\{x_n\}_n$ in P converges to $x \in P$ i.e., iff $\lim_{x \to \infty} \mu(x_n, x, t) = 1$ $\Rightarrow x \in P$.

Definition 2.8: Let $(X, \mu, *)$ be a fuzzy metric space, $x \in X$, $r \in (0,1)$, $t > 0$, $S(x, r, t) = \{ y \in X / \mu(x, y, t) > 1 - r \}$. Hence $S(x, r, t)$ is said to be a closed ball centered at x of radius r with respect to t iff. Any sequence $\{x_n\}$ in $S(x, r, t)$ converges to some $y \in S(x, r, t)$.

Definition 2.9: A sequence $\{x_n\}$ in fuzzy metric space is said to converge to $x \in X$ if and only if $\lim_{x \to \infty} \mu(x_n, y, t) = 1.$

A sequence $\{x_n\}$ in fuzzy metric space is said to be a Cauchy sequence if and only if $\lim_{x \to \infty} \mu(x_{n+p}, x_n, t) = 1.$

A fuzzy metric space $(X, \mu, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Lemma 2.10 [6]: Let $(X, \mu, *)$ be fuzzy metric space. If $x_n \to x$ and $y_n \to y$ in $(X, \mu, *)$ then $\mu(x_n, y_n, t) \to \mu(x, y, t)$ as $n \to \infty$ for all $t > 0$ in $\mathbb{R}$, the set of all real numbers.

Lemma 2.11: Let $x \in X$, $A \in W(X)$ and $\{x\}$ be a fuzzy set with membership function equal a characteristic function of set $\{x\}$. If $x_\alpha \subset A$ if and only if $P_\alpha(x, A, t) = 1$ for each $\alpha \in [0, 1]$.

Proof: If $x_\alpha \subset A$ then $x \in A_\alpha$ for each $\alpha \in [0,1]$, $P_\alpha(x, A, t) = \sup_{y \in A_\alpha} \mu(x, y, t) = 1$.

Conversely, if $P_\alpha(x, A, t) = 1$, then $\sup_{y \in A_\alpha} \mu(x, y, t) = 1$. It follows that $x \in \bar{A}_\alpha = A_\alpha$ for each $\alpha \in [0, 1]$. Thus $x_\alpha \subset A$.

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Lemma 2.12: $P_\alpha(x, A, t) \geq \mu\left(x, y, \frac{t}{2}\right) * P_\alpha\left(y, A, \frac{t}{2}\right) \quad \forall\, x, y \in X$

Proof: $P_\alpha(x, A, t) = \sup_{z \in A_\alpha} \mu(x, z, t) \geq \sup_{z \in A_\alpha}\left(\mu\left(x, y, \frac{t}{2}\right) * \mu\left(y, z, \frac{t}{2}\right)\right)$

$$= \mu\left(x, y, \frac{t}{2}\right) * P_\alpha\left(y, A, \frac{t}{2}\right)$$

Lemma 2.13: If $\{x_0\} \subset A$, then $P_\alpha(x_0, B, t) \geq D_\alpha(A, B, t)$ for each $B \in W(X)$

Proof: $P_\alpha(x_0, B, t) = \sup_{y \in B_\alpha} \mu(x_0, y, t)$

$$\geq \inf_{x \in A_\alpha} \sup_{z \in B_\alpha} \mu(x, y, t) \geq D_\alpha(A, B, t)$$

MAIN RESULTS

Theorem 3.1: Let $(X, \mu, *)$ be a complete fuzzy metric space and F be a continuous fuzzy mapping from X to $W_\alpha(X)$ satisfying the following condition:

$$D_\alpha(F(x), F(y), kt) \geq \min\{\mu(x, y, t), P_\alpha(x, F(x), t), P_\alpha(y, F(y), t),$$
$$P_\alpha(x, F(y), t), P_\alpha(y, F(x), t)\} \qquad (1)$$

for all $x, y \in X$, $\alpha \in (0, 1]$ and $k \in \left(0, \frac{1}{4}\right)$.

Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F.

Proof: Let $x_0 \in X$ suppose that there exists $x_1 \in (F(x_0))_\alpha$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \in (F(x_1))_\alpha$ and by lemma **2.13**

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we get, $\mu\left(x_1, x_2, k\,t\right) = P_\alpha\left(x_1, F\left(x_1\right), k\,t\right) \geq D_\alpha\left(F\left(x_0\right), F\left(x_1\right), k\,t\right)$

By induction we construct a sequence $\left\{x_n\right\}$ in X such that $x_n \in \left(F\left(x_{n-1}\right)\right)_\alpha$ and

$$\mu\left(x_n, x_{n+1}, k\,t\right) = P_\alpha\left(x_n, F\left(x_n\right), k\,t\right) \geq D_\alpha\left(F\left(x_{n-1}\right), F\left(x_n\right), k\,t\right)$$

$$\geq \min\left\{\mu\left(x_{n-1}, x_n, t\right), P_\alpha\left(x_{n-1}, F\left(x_{n-1}\right), t\right), P_\alpha\left(x_n, F\left(x_n\right), t\right),\right.$$

$$\left. P_\alpha\left(x_{n-1}, F\left(x_n\right), t\right), P_\alpha\left(x_n, F\left(x_{n-1}\right), t\right)\right\}$$

$$\geq \min\left\{\mu\left(x_{n-1}, x_n, t\right), \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * P_\alpha\left(x_n, F\left(x_{n-1}\right), \frac{t}{2}\right),\right.$$

$$\mu\left(x_n, x_{n+1}, \frac{t}{2}\right) * P_\alpha\left(x_{n+1}, F\left(x_n\right), \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) *$$

$$\left. P_\alpha\left(x_n, F\left(x_n\right), \frac{t}{2}\right), P_\alpha\left(x_n, F\left(x_{n-1}\right), t\right)\right\}$$

$$\geq \min\left\{\mu\left(x_{n-1}, x_n, t\right), \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * 1, \mu\left(x_n, x_{n+1}, \frac{t}{2}\right) * 1,\right.$$

$$\left. \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4}\right) * P_\alpha\left(x_{n+1}, F\left(x_n\right), \frac{t}{4}\right), 1\right\}$$

$$\geq \min\left\{\mu\left(x_{n-1}, x_n, t\right), \mu\left(x_{n-1}, x_n, \frac{t}{2}\right), \mu\left(x_n, x_{n+1}, \frac{t}{2}\right),\right.$$

$$\left. \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4}\right) * 1, 1\right\}$$

$$\geq \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4}\right)$$

which implies that

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$$\mu\left(x_n, x_{n+1}, kt\right) \geq \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4}\right)$$

$$\Rightarrow \mu\left(x_n, x_{n+1}, t\right) \geq \mu\left(x_{n-1}, x_n, \frac{t}{2k}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4k}\right)$$

$$\vdots$$

$$\geq \mu\left(x_{n-1}, x_n, \frac{t}{(2k)^n}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{(4k)^n}\right)$$

$$\geq 1 * 1 = 1$$

$$\Rightarrow \mu\left(x_n, x_{n+1}, t\right) = 1$$

We now verify that $\{x_n\}$ is a Cauchy sequence in $(X, \mu, *)$. Let $t_1 = \frac{t}{p}$.

$$\mu\left(x_n, x_{n+p}, t\right) \geq \mu\left(x_n, x_{n+1}, t_1\right) * \mu\left(x_{n+1}, x_{n+2}, t_1\right) * \cdots * \mu\left(x_{n+p-1}, x_{n+p}, t_1\right)$$

$$\Rightarrow \mu\left(x_n, x_{n+p}, t\right) = 1$$

$\Rightarrow \{x_n\}_n$ is Cauchy sequence in $(X, \mu, *)$.

Since X is a complete, there exists $x \in X$ such that $x_n \to x$ in $(X, \mu, *)$.

Now by Lemmas **2.12** and **2.13** we have

$$P_\alpha\left(x, F(x), kt\right) \geq \mu\left(x, x_n, \frac{kt}{2}\right) * P_\alpha\left(x_n, F(x), \frac{kt}{2}\right)$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * D_\alpha\left(F\left(x_{n-1}\right), F(x), \frac{kt}{2}\right)$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * \min\left\{\mu\left(x_{n-1}, x, \frac{t}{2}\right), P_\alpha\left(x_{n-1}, F\left(x_{n-1}\right), \frac{t}{2}\right),\right.$$

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$$P_\alpha\left(x, F(x), \frac{t}{2}\right), P_\alpha\left(x_{n-1}, F(x), \frac{t}{2}\right), P_\alpha\left(x, F\left(x_{n-1}\right), \frac{t}{2}\right)\Big\}$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * \min\Big\{\mu\left(x_{n-1}, x, \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{4}\right) *$$

$$P_\alpha\left(x_n, F\left(x_{n-1}\right), \frac{t}{4}\right), P_\alpha\left(x, F(x), \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{4}\right) *$$

$$P_\alpha\left(x_n, F(x), \frac{t}{4}\right), \mu\left(x, x_n, \frac{t}{4}\right) * P_\alpha\left(x_n, F\left(x_{n-1}\right), \frac{t}{4}\right)\Big\}$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * \min\Big\{\mu\left(x_{n-1}, x, \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{4}\right) * 1,$$

$$P_\alpha\left(x, F(x), \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{4}\right) * \mu\left(x_n, x, \frac{t}{8}\right) * P_\alpha\left(x, F(x), \frac{t}{4}\right),$$

$$\mu\left(x, x_n, \frac{t}{4}\right) * 1\Big\}$$

taking limit as $n \to \infty$, we have

$$\Rightarrow P_\alpha(x, F(x), t) \geq P_\alpha\left(x, F(x), \frac{t}{4k}\right) \geq \cdots \geq P_\alpha\left(x, F(x), \frac{t}{(4k)^n}\right)$$

$\Rightarrow P_\alpha(x, F(x), t) = 1$ and by lemma **2.11**, $x_\alpha \subset F(x)$.

This completes the proof.

Example. Let $X = [0,1]$ and $\mu : X \times X \to \mathbb{R}^+$ where $\mu(x, y, t) = \dfrac{t}{t + |x - y|}$ for $x, y \in X$. Let $\alpha \in \left(0, \frac{1}{2}\right)$ and suppose $F : X \to I^X$ defined by

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$$F(0)(x)=\begin{cases}1 & x=0\\ \alpha & x\in\left(0,\frac{1}{2}\right]\\ \frac{\alpha}{2} & x\in\left(\frac{1}{2},1\right]\end{cases}\qquad F(1)(x)=\begin{cases}1 & x=0\\ 2\alpha & x\in\left(0,\frac{1}{2}\right]\\ \frac{\alpha}{2} & x\in\left(\frac{1}{2},1\right]\end{cases}$$

and for $z\in(0,1)$.

$$F(z)(x)=\begin{cases}1 & x=0\\ 2\alpha & x\in\left(0,\frac{1}{2}\right]\\ 0 & x\in\left(\frac{1}{2},1\right]\end{cases}$$

Then, $F(0)_1=F(z)_1=F(1)_1=\{0\}\ F(0)_\alpha=F(z)_\alpha=F(1)_\alpha=\left[0,\frac{1}{2}\right]$

and $F(0)_{\frac{\alpha}{2}}=F(1)_{\frac{\alpha}{2}}=[0,1],\ F(z)_{\frac{\alpha}{2}}=\left[0,\frac{1}{2}\right]$

Consequently,

$$P_1(F(x),F(y),kt)=\sup\left\{\frac{kt}{kt+|x-y|}\text{ for } x\in F(x)_1, y\in F(y)_1\right\}\text{ and}$$

$$D_1(F(x),F(y),kt)=H\left(F(x)_1,F(y)_1,kt\right)$$

$$=\inf\left\{\inf_{x\in F(y)_1}\ \sup_{y\in F(x)_1}\frac{kt}{kt+|x-y|},\ \inf_{x\in F(x)_1}\ \sup_{y\in F(y)_1}\frac{kt}{kt+|x-y|}\right\}$$

Then LHS of (1), $D_1(F(x),F(y),kt)=1\quad\forall x,y\in X$.

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and for all $x, y \in X$ the RHS of (1),

$$\mu(x , y, t) < 1, \; P_1(x, F(x), t) \leq 1, \; P_1(y, F(y), kt) \leq 1,$$

$$P_1(x , F(y), t) \leq 1, \; P_1(y, F(x), t) \leq 1,$$

and therefore

$$\min\{ \mu(x, y, t), P_1(x, F(x), t), P_1(y, F(y), t), P_1(x, F(y), t), P_1(y, F(x), t) \} < 1.$$

Thus, (1) holds.

$$P_\alpha(F(x), F(y), kt) = \sup\left\{ \frac{kt}{kt + | x - y |} \text{ for } x \in F(x)_\alpha , y \in F(y)_\alpha \right\}$$

$$D_\alpha(F(x), F(y), kt) = H\left(F(x)_\alpha , F(y)_\alpha , kt \right)$$

$$= \inf\left\{ \inf_{x \in F(y)_\alpha} \sup_{y \in F(x)_\alpha} \frac{kt}{kt + | x - y |} , \inf_{x \in F(x)_\alpha} \sup_{y \in F(y)_\alpha} \frac{kt}{kt + | x - y |} \right\}$$

Now, LHS of (1), $D_\alpha(F(x), F(y), kt) = 1 \quad \forall \; x, \; y \in X$ and the RHS of (1),

$$\min\{ \mu(x, y, t), P_\alpha(x, F(x), t), P_\alpha(y, F(y), t), P_\alpha(x, F(y), t), P_\alpha(y, F(x), t) \} < 1.$$

Hence, (1) holds. Again, we see that

$$P_{\frac{\alpha}{2}}(F(x), F(y), kt) = \sup\left\{ \frac{kt}{kt + | x - y |} \text{ for } x \in F(x)_{\frac{\alpha}{2}}, y \in F(y)_{\frac{\alpha}{2}} \right\}$$

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$$D_{\frac{\alpha}{2}}\left(F(x),F(y),\,kt\right)=H\left(F(x)_{\frac{\alpha}{2}},\,F(y)_{\frac{\alpha}{2}},kt\right)$$

$$=\inf\left\{\inf_{x\in F(y)_{\frac{\alpha}{2}}}\ \sup_{y\in F(x)_{\frac{\alpha}{2}}}\frac{kt}{kt+|x-y|},\ \inf_{x\in F(x)_{\frac{\alpha}{2}}}\ \sup_{y\in F(y)_{\frac{\alpha}{2}}}\frac{kt}{kt+|x-y|}\right\}$$

Now, the LHS of (1),

$D_{\frac{\alpha}{2}}\left(F(x),\,F(y),\,kt\right)=1 \quad \forall\ x,\ y\in X$ and the RHS of (1),

$$\min\left\{\mu(x,\,y,\,t),\,P_{\frac{\alpha}{2}}(x,F(x),\,t),\,P_{\frac{\alpha}{2}}(y,F(y),t),\,P_{\frac{\alpha}{2}}(x,F(y),t),\right.$$

$$\left.P_{\frac{\alpha}{2}}(y,F(x),t)\right\}<1.$$

Hence, (1) holds.

Thus (1) holds and hence all the conditions of the Theorem (3.1) are satisfied.

Applying the Theorem (**3.1**), we can conclude that F has a fixed fuzzy point in X .

Corollary 3.2: Let $(X,\,\mu,\,*)$ be a complete fuzzy metric space. Let F be a fuzzy mapping from X in to $W_{\alpha}(X)$ satisfying the following condition:

There exists $\alpha\in(0,1]$ and $k\in(0,1)$ such that

$$k\,D_{\alpha}\left(F(x),\,F(y),\,t\right)\geq\mu(x,\,y,\,t) \qquad (1)$$

for all $x,y\in X$, $\alpha\in(0,1]$ and $k\in\left(0,\frac{1}{4}\right)$.

Then there exists $x\in X$ such that x_{α} is a fixed fuzzy point of F .

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Theorem 3.3: Let $\alpha \in (0,1]$, $k \in (0,1)$ and $(X, \mu, *)$ be a complete fuzzy metric space. Let F be a fuzzy mapping from X in to $W_\alpha(X)$ satisfying the condition (1) when $x, y \in S(x, r, t)$. Moreover, assume that

$$k\,P_\alpha(x, F(x), t) > (1-r) \qquad \forall\, x \in X.$$

Then F has fixed fuzzy point in $S(x, r, t)$.

Proof: $x_1 \in (F(x))_\alpha$, $x_2 \in (F(x_1))_\alpha, \cdots, x_n \in (F(x_{n-1}))_\alpha$ for all $n \in N$. Now

$$k\,P_\alpha(x, F(x), t) > (1-r) \Rightarrow P_\alpha(x, F(x), t) > \frac{(1-r)}{k} > (1-r)$$

$$\Rightarrow P_\alpha(x, F(x), t) > (1-r)$$

$$\Rightarrow \mu(x, x_1, t) = P_\alpha(x, F(x), t) > (1-r)$$

$$\Rightarrow \mu(x, x_1, t) > (1-r)$$

$$\Rightarrow x_1 \in S(x, r, t)$$

Assuming that $x_1, x_2, x_3, \cdots, x_{n-1} \in S(x, r, t)$. We show that $x_n \in S(x, r, t)$

$$k\,P_\alpha(x_1, F(x_1), t) \geq k\,D_\alpha(F(x), F(x_1), t) \geq \mu(x, x_1, t) > (1-r)$$

$$\Rightarrow P_\alpha(x_1, F(x_1), t) > \frac{(1-r)}{k} > (1-r)$$

$$\Rightarrow \mu(x_1, x_2, t) > (1-r)$$

Again,

$$k\,P_\alpha(x_2, F(x_2), t) \geq k\,D_\alpha(F(x_1), F(x_2), t) \geq \mu(x_1, x_2, t) > (1-r)$$

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$$\Rightarrow P_{\alpha}\left(x_2, F\left(x_2\right), t\right) > \frac{\left(1-r\right)}{k} > \left(1-r\right)$$

$$\Rightarrow \mu\left(x_2, x_3, t\right) > \left(1-r\right)$$

Similarly, it can be shown that,

$$P_{\alpha}\left(x_3, F\left(x_3\right), t\right) > \left(1-r\right), \cdots, P_{\alpha}\left(x_n, F\left(x_n\right), t\right) > \left(1-r\right)$$

Let $t_1 = \frac{t}{p}$. Thus, we see that,

$$\mu\left(x, x_n, t\right)$$

$$\geq \mu\left(x, x_1, \frac{t}{n}\right) * \mu\left(x_1, x_2, \frac{t}{n}\right) * \cdots * \mu\left(x_{n-1}, x_n, \frac{t}{n}\right)$$

$$> \left(1-r\right) * \left(1-r\right) * \cdots * \left(1-r\right) > \left(1-r\right)$$

$$\Rightarrow \mu\left(x, x_n, t\right) > \left(1-r\right) \Rightarrow x_n \in S\left(x, r, t\right)$$

From the lemmas **2.11**, **2.12** and **2.13**, we can say that the rest of the proof is obvious.

Therefore, $x_{\alpha} \subset F\left(x\right)$.

This completes the proof.

Theorem 3.4: Let $\left(X, \mu, *\right)$ be a complete fuzzy metric space. Let F and G be continuous fuzzy mappings from X to $W_{\alpha}\left(X\right)$ satisfying the following condition:

$$D_{\alpha}\left(F(x), G(y), kt\right) \geq \min\left\{\mu\left(x, y, t\right), P_{\alpha}\left(x, F(x), t\right), P_{\alpha}\left(y, G(y), t\right), P_{\alpha}\left(x, G(y), t\right), P_{\alpha}\left(y, F(x), t\right)\right\}$$

for all $x, y \in X$, $\alpha \in \left(0, 1\right]$ and $k \in \left(0, \frac{1}{4}\right)$.

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Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F, G.

Proof: Let $x_0 \in X$, Since $\left(F\left(x_0\right)\right)_\alpha$ is nonempty subset of X, then there exists $x_1 \in \left(F\left(x_0\right)\right)_\alpha$. Then there exists $x_2 \in X$ such that $x_2 \in \left(F\left(x_1\right)\right)_\alpha$ also since $\left(G\left(x_1\right)\right)_\alpha$ is nonempty subset of X, then there exists $x_2 \in \left(G\left(x_1\right)\right)_\alpha$ and by lemma **2.13**, we get

$$\mu\left(x_1, x_2, k\,t\right) = P_\alpha\left(x_1, G\left(x_1\right), k\,t\right) \geq D_\alpha\left(F\left(x_0\right), G\left(x_1\right), k\,t\right)$$

$$\geq \min\left\{\mu\left(x_0, x_1, t\right), P_\alpha\left(x_0, F\left(x_0\right), t\right), P_\alpha\left(x_1, G\left(x_1\right), t\right), P_\alpha\left(x_0, G\left(x_1\right), t\right), P_\alpha\left(x_1, F\left(x_0\right), t\right)\right\}$$

$$\geq \min\left\{\mu\left(x_0, x_1, t\right), \mu\left(x_0, x_1, \frac{t}{2}\right) * P_\alpha\left(x_1, F\left(x_0\right), \frac{t}{2}\right), \mu\left(x_1, x_2, \frac{t}{2}\right) * P_\alpha\left(x_2, G\left(x_1\right), \frac{t}{2}\right), \mu\left(x_0, x_1, \frac{t}{2}\right) * P_\alpha\left(x_1, G\left(x_1\right), \frac{t}{2}\right), 1\right\}$$

$$\geq \min\left\{\mu\left(x_0, x_1, t\right), \mu\left(x_0, x_1, \frac{t}{2}\right) * 1, \mu\left(x_1, x_2, \frac{t}{2}\right) * 1, \mu\left(x_0, x_1, \frac{t}{2}\right) * \mu\left(x_1, x_2, \frac{t}{4}\right) * 1, 1\right\}$$

$$\geq \mu\left(x_0, x_1, \frac{t}{2}\right) * \mu\left(x_1, x_2, \frac{t}{4}\right)$$

By induction we construct a sequence $\left\{x_n\right\}$ in X such that $x_n \in \left(F\left(x_{n-1}\right)\right)_\alpha$, $x_{n+1} \in \left(G\left(x_n\right)\right)_\alpha$ and

$$\mu\left(x_n, x_{n+1}, k\,t\right) = P_\alpha\left(x_n, G\left(x_n\right), k\,t\right) \geq D_\alpha\left(F\left(x_{n-1}\right), G\left(x_n\right), k\,t\right)$$

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$$\geq \min\Big\{ \mu(x_{n-1}, x_n, t),\ P_\alpha(x_{n-1}, F(x_{n-1}), t), P_\alpha(x_n, G(x_n), t),$$

$$P_\alpha(x_{n-1}, G(x_n), t),\ P_\alpha(x_n, F(x_{n-1}), t)\Big\}$$

$$\Rightarrow \mu(x_n, x_{n+1}, kt) \geq \mu\left(x_{n-1}, x_n, \frac{t}{2}\right) * \mu\left(x_n, x_{n+1}, \frac{t}{4}\right)$$

As in the above theorem **3.1**, the proof is similar.

$\{x_n\}_n$ is Cauchy sequence in $(X, \mu, *)$.

Since X is a complete, there exists $x \in X$ such that $x_n \to x$ in $(X, \mu, *)$.

Now by Lemmas **2.12** and **2.13** we have

$$P_\alpha(x, G(x), kt) \geq \mu\left(x, x_n, \frac{kt}{2}\right) * P_\alpha\left(x_n, G(x), \frac{kt}{2}\right)$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * D_\alpha\left(F(x_{n-1}), G(x), \frac{kt}{2}\right)$$

$$\geq \mu\left(x, x_n, \frac{kt}{2}\right) * \min\Big\{\mu\left(x_{n-1}, x, \frac{t}{2}\right), \mu\left(x_{n-1}, x_n, \frac{t}{4}\right) * 1,\ P_\alpha\left(x, G(x), \frac{t}{2}\right),$$

$$\mu\left(x_{n-1}, x_n, \frac{t}{4}\right) * \mu\left(x_n, x, \frac{t}{8}\right) * P_\alpha\left(x, G(x), \frac{t}{4}\right), \mu\left(x, x_n, \frac{t}{4}\right) * 1\Big\}$$

taking limit as $n \to \infty$, we have

$$\Rightarrow P_\alpha(x, G(x), t) \geq P_\alpha\left(x, G(x), \frac{t}{4k}\right) \geq \cdots \geq P_\alpha\left(x, G(x), \frac{t}{(4k)^n}\right)$$

$\Rightarrow P_\alpha(x, G(x), t) = 1$ and by lemma **2.11**, $x_\alpha \subset G(x)$. Similarly, $x_\alpha \subset F(x)$.

This completes the proof.

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