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Original Research Article

The Beta-Power Half-Cauchy Exponential Distribution: Model Properties and an Application to Fatigue Data

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Abstract

This paper introduces the BetaPower HalfCauchy Exponential (BPHCE) distribution, a new twoparameter lifetime model derived by incorporating a power transformation into the halfCauchy exponential framework. The distribution offers enhanced flexibility for modeling positive data with heavytailed and skewed characteristics, while retaining a closedform cumulative distribution function. We derive essential statistical properties including the Probability density function(pdf), Hazard Rate function(hrf), and quantile function.

Parameters of the distribution are estimated using maximum likelihood. The proposed distribution is applied to the classic fatiguelife dataset to check the practical applicability of the distribution.

Model comparison based on AIC, BIC, and related criteria shows that the BPHCE distribution outperforms several competing lifetime models, such as exponentiated Weibull, generalized exponential, and Weibull distributions. The results confirm that the BPHCE is a valuable addition to the toolbox of reliability analysts, capable of effectively describing realworld lifetime data with complex tail behavior. The generated model will be more applicable in data analysis of various field and will help the researcher in generating new probability models and applying on the real data sets available in modern real life.

Introduction

Background

Probability distribution plays a fundamental contribution in statistical modeling of diverse scientific discipline such as engineering, environmental science, finance, reliability and survival analysis. In modeling, the choice of appropriate probability model is very crucial to get accurate and precise representation of data, prediction and inferences. Classical models like the normal, Weibull, exponential, and gamma are backbone of statistical practice. But in real-world problem which contains data complexities such as skewed, heavy tailed, or multimodality are not adequately captured by classical models.

In modern statistical literatures, there has been various families of distributions helping to generalizing the classical models to generate more flexible probability model capturing various in real data sets. Common approaches that help in formulation of new probability models are; transformation methods which applies power transformations & log-transformations; composite or mixed models that combine two or more distributions to generate new models; generated families which uses generators like the gamma-g, beta-G etc. and truncated methods which generates models like half-normal & half-Cauchy

For modeling the positive data with heavy-tailed behaviors, the half -Cauchy distribution is found to be more appropriate choice of the researchers. It is derived by folding the standard Cauchy distribution at zero providing a very useful probability model for various applications specially in Bayesian statistics as a prior for scale parameters. Building upon this foundation of half - Cauchy distribution, various extensions have been proposed to generate new models. The half-Cauchy exponential (HCE) distribution formulated using power transformation offers greater flexibility in modeling of lifetime data analysis.

Previous Studies

The Cauchy distribution was first studied by Poisson (1824) and later by Cauchy (1853), is a classical symmetric heavy-tailed probability distribution. Johnson et al. (1994) studied it in detailed covering the properties. The half-Cauchy distribution was obtained by truncating the Cauchy distribution at zero and multiplying by 2 which was used in Bayesian statistics as a prior for scale parameters. Gelman (2006) advocated for the half-Cauchy as a default prior noting better compared to inverse -gamma priors. Cooray and Ananda (2008) generated a half-normal distribution for lifetime data. Brazauskas and Kleefeld (2011) used heavy-tailed distributions including Cauchy variants for modeling Norwegian insurance loss data.

The generator methods have become a popular method for generating more flexible model latter. Eugene et al. (2002) proposed the beta-generated family which paved the path for numerous subsequent families such as gamma-G (Zografos and Balakrishnan, 2009), the Kumaraswamy-G (Corderio and de Castro, 2011) and the logistic-X (Tahir et al., 2016) were proposed. Alshawarbeh et al. (2013) introduced the beta-Cauchy which uses Cauchy based proposed families.

Shaw and Buckley (2009) studied general transformation techniques on generating skew distributions. During Box and Cox's (1964) seminal work on transformation to normality, the power transformations for generating new probability model was employed. Gupta and Gupta (2008) studied the power- normal distribution to analyze skewed data.

The searching of more flexible and precise lifetime distributions continues in reliability engineering since long time. Mudholkar and Srivastava (1993) generated the exponentiated Weibull, while Gupta and Kundu (1999) gave generalized exponential model. Cordeiro et al. (2013) provided a comprehensive review of the beta-generated distributions in lifetime data analysis. In recent extension, Telee and Kumar (2019) proposed a more flexible model as extension of half-Cauchy distribution and Chaudhary et al. (2024) proposed a New Extended Kumaraswamy Exponential Distribution, more flexible distribution which addresses heavy-tailed characteristics in environmental data

Research Gap

Although numerous extensions of Cauchy distributions are accessible in theory and power transformations have been applied to other families, the power transformation within a half-Cauchy exponential framework appears not to have been studied broadly. The proposed distribution is able to fill this gap by contributing a flexible two-parameter probability model having closed-form cumulative distribution function, more precisely suitable for positive real data with inconsistent tail behavior.

Beta-Power half-Cauchy exponential (BPHCE) distribution

This study, introduces two parameter Beta-Power half-Cauchy exponential (BPHCE) distribution that incorporates a power transformation, substantially enhancing its modeling capability. The cdf of the BPHCE is:

$$F(x;\lambda,\beta)=\frac{2}{\pi}\arctan\left(x^{\beta}e^{-\lambda/x}\right),\quad x>0,\lambda>0,\beta>0 \tag{1}$$

Here, scale parameter λ control the exponential decay rate, while another parameter β is shape parameter that govern the power-law behavior in the lower tail of the model. The additional shape parameter β allows the distribution to model various tail behaviors and skewness patterns and it maintains a closed-form CDF adding flexibility than its predecessor.

Special cases

Case 1: $\lambda = 0$

If $\lambda = 0$ then, $e^{-\lambda/x} = 1$, so

$$F(x;\beta) = \frac{2}{\pi}\arctan\left(x^{\beta}\right)$$

This is the power half-Cauchy distribution or half -Cauchy with power transformation.

Case 2: $\beta = 1, \lambda = 0$

$$F(x;1) = \frac{2}{\pi}\arctan\left(x\right)$$

This is standard half-Cauchy distribution with scale = 1

The corresponding pdf of the BPHCE is:

$$f(x;\lambda,\beta) = \frac{2}{\pi}e^{-\lambda/x}x^{\beta-2}\left(\beta+\frac{\lambda}{x}\right)\left(1+x^{2\beta}e^{-2\lambda/x}\right)^{-1}, \quad x>0 \tag{2}$$

The reliability and hazard rate function of the models are mentioned in the expression (3) and (4) respectively

$$R(x) = 1-\frac{2}{\pi}\arctan\left(x^{\beta}e^{-\lambda/x}\right), \text{ and} \tag{3}$$

$$h(x) = \frac{2}{\pi}e^{-\lambda/x}x^{\beta-2}\left(\beta+\frac{\lambda}{x}\right)\left(1+x^{2\beta}e^{-2\lambda/x}\right)^{-1}\left[1-\frac{2}{\pi}\arctan\left(x^{\beta}e^{-\lambda/x}\right)\right]^{-1} \tag{4}$$

To demonstrate the variations in shape of the model, for various sets of the parameter, different shaped pdf curves (left) and the hazard rate curve(right) are demonstrated in figure 1. These variations in shape of the pdfs clearly confirms that the proposed model will represent the datasets of different nature and fields. Also, the nature of the pdf curves demonstrates that the proposed distribution is non normal and unimodal in nature.

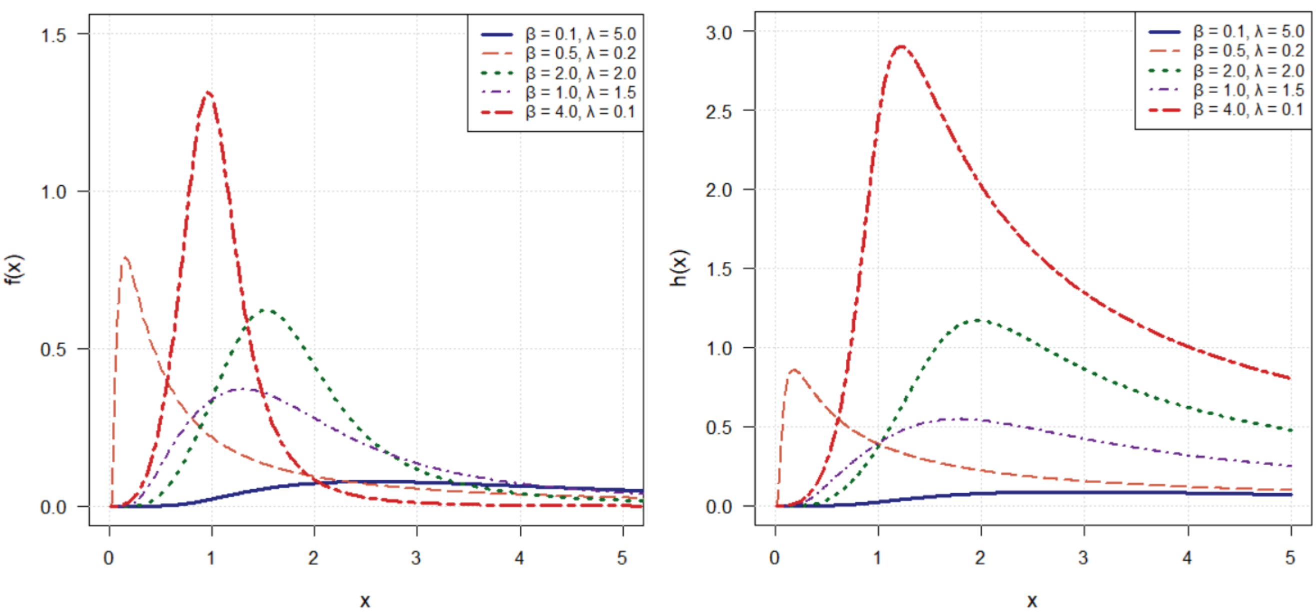

Figure 1: *Densities plots (Left) and Hazard rate plots (Right) of BPHCE*

Quantile Function

The quantile function which helps to derive the quantile values of the probability distribution has very important role in model validation as well as demonstrating the nature of the probability distributions. The quantile function of BPHCE is:

$$x_p^{\beta} \exp\left(\frac{-\lambda}{x_p}\right) = \tan\left(\frac{\pi p}{2}\right), \quad 0 < p < 1 \tag{5}$$

This is a transcendental equation requiring numerical methods to solve for x_p

Parameter Estimations using Maximum Likelihood estimation

The likelihood function is the key expression for the probability distribution. It has very important role in parameter estimation in modeling of the probability distribution and model validation check. The Log-Likelihood function of the BPHCE is given by expression (6)

$$\ell(\lambda,\beta \mid x) = n\ln\left(\frac{2}{\pi}\right) + (\beta-1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\ln\left(\beta+\frac{\lambda}{x_i}\right) - \sum_{i=1}^{n}\ln\left[1+x_i^{2\beta}\exp\left(\frac{-2\lambda}{x_i}\right)\right] \tag{6}$$

The MLE is based on the optimization theory to estimate the parameters. For optimization, we find the partial as well as the cross partial derivatives. The MLEs $(\hat{\lambda},\hat{\beta})$ satisfy:

$$\frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \beta} = 0$$

Partial derivatives are:

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{(1/x_i)}{\beta+\frac{\lambda}{x_i}} - 2\sum_{i=1}^{n}\frac{x_i^{2\beta-1}\exp\left(-\frac{2\lambda}{x_i}\right)}{1+x_i^{2\beta}\exp\left(-\frac{2\lambda}{x_i}\right)}$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{1}{\beta+\frac{\lambda}{x_i}} - 2\sum_{i=1}^{n}\frac{x_i^{2\beta}\ln x_i\exp\left(-\frac{2\lambda}{x_i}\right)}{1+x_i^{2\beta}\exp\left(-\frac{2\lambda}{x_i}\right)}$$

The observed Fisher information matrix is:

$$I(\beta,\lambda) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \lambda} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}$$

Since the derivatives are nonlinear in nature, so will be quite impossible to solve the derivatives analytically, so here, numerical method will be applicable for estimating the unknown parameters for model fitting.

Application to a real dataset

To evaluate the practical usefulness of the BPHCE model, it is applied to an actual data set. This data set describes the fatigue-life of 6061-T6 aluminum coupons that were cut to the rolling direction. The specimens were subjected to cyclic loading with frequency of 18 cycles per seconds as reported by Birnbaum and Saunders (1969). The data consists of 101 observations; each recorded under a maximum cycle stress level of 31,000 psi.

104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 133, 134, 134, 148, 148, 149, 70, 90, 96, 97, 99, 100, 103, 159, 162, 163, 163, 164, 166,166, 168, 170, 174, 196, 212, 156, 157, 157, 157, 157, 158,121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132,134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146,151, 151, 152, 155

The summary of the data displayed in table 1 show it as nonnormal with positive skewness.

Table1: *Descriptive Statistics of the dataset*

n	Min.	Mean	Median	SD	Q_1	Q_3	Ma.	SK	Kurtosis
101	70	133.73	133	22.35	120	146	212	0.074	1.086

As the estimation of the parameter is impossible analytically, we have used the R programming (R Core Team, 2024) For analysis of the data, we have used the Optim () function as well as the package NeuDist (Kumar et al., 2025), The MLE of data are given in table 2.

Table 2: *Maximum Likelihood Estimates*

Parameters	Estimates	SE	95% C.I.
β	1.5425	0.139532	[1.269022, 1.815978]
λ	999.9612	90.05595	[823.4547, 1176.468]

Loglikelihood values and various information criterion which helps in model comparison and model validation are mentioned in table 3. These information criteria are Akaike's Information Criteria (AIC), Bayesian information Criteria (BIC), Consistent Akaike's Information Criterion (CAIC) as well as Hannan-Quinn Information Criterion (HQIC). We have also tested the goodness-of-fit of the model using Kolmogorov-Smirnov test. The test statistics value of 0.073 and the corresponding p value of 0.648(> 0.05) demonstrate significance of the model.

Table 3: *Maximum Likelihood Estimates and Model Fit Statistics*

LL	AIC	BIC	CAIC	HQIC	KS(p-value)
-458.3888	920.7775	926.0078	920.8999	922.8949	0.073(0.648)

Model validation and Comparisons

Testing of the model validation is the key part of the modeling. there are various analytical and the graphical methods available in theory that helps in validation checking of the fitted model based on datasets. one of the graphical methods of demonstrating the P-P and Q-Q plots which are displayed in figure 2. Figure demonstrate that the proposed model BPHCE fits the data precisely.

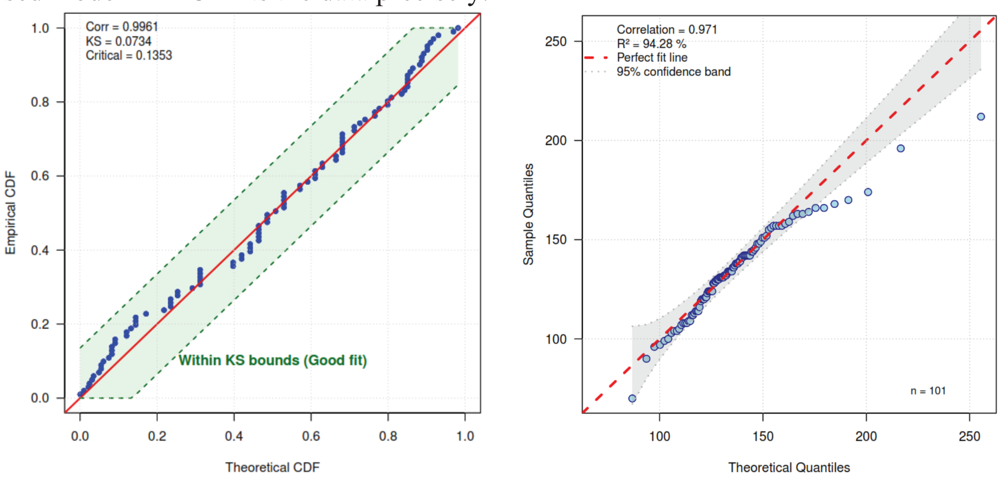

Figure 2: *The P-P plots (Left) and Q-Q plots (Right) of the BPHCE model.*

Also, to show the accuracy of the observed data and the fitted data, we have displayed histogram versus fitted pdf of the BPHCE along with the empirical cdf versus fitted cdf and are displayed in (figure 3) which also helps in testing of the model validation. Kolmogorov -Smirnov test statistics of 0.03 with p value of 0.648 supports validity of the model for given set of the data.

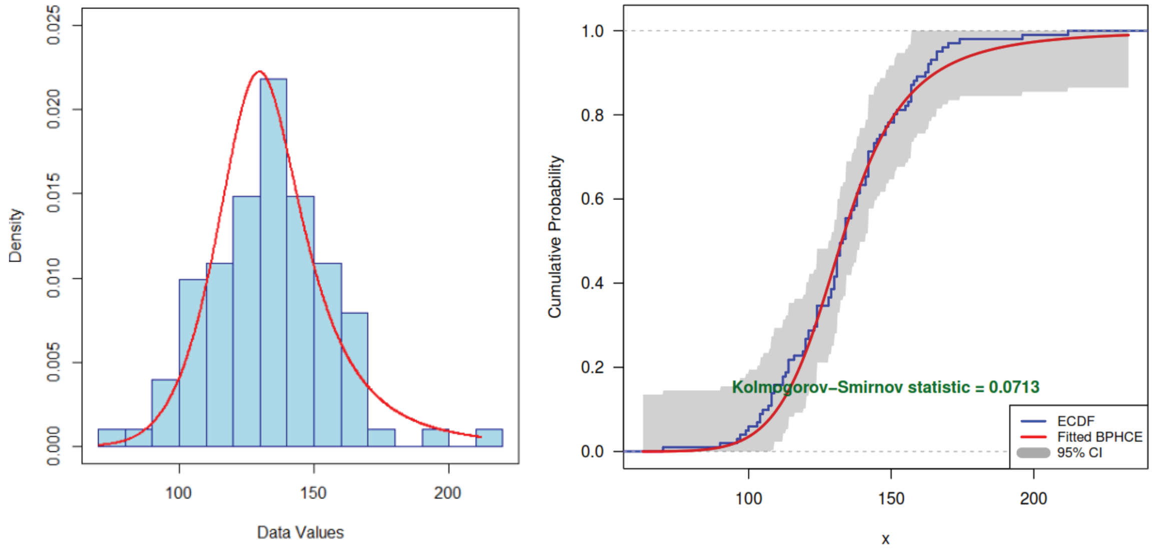

Figure 3: *Fitted pdf vs Hist. (left) and ECDF vs. fitted CDF (right) of the BPHCE*

To verify the superiority of the model, formulated BPHCE model is compared with seven other probability models already published and available in theory. The competing models are half Cauchy Extended Exponential (HCEE)(Chaudhary et al., 2022), Chen distribution proposed by Chen, (2000), Exponential Extension(EE) distribution (Nadarajah & Haghighi, 2011), Modified Weibull(MW) distribution(Lai et al., 2003), Generalized Exponential(GE) distribution(Gupta & Kundu, 1999), Weibull Extension (WE) model(Tang et al., 2003), and Exponentiated Weibull(EW) (Mudholakar & Srivastav, 1993) The loglikelihood values along with the various information criterion demonstrate superior among competitive models

Table 5: *The information criterion of the BPHCE and Competing Models*

Model	-LL	AIC	BIC	CAIC	HQIC
BPHCE	458.40	920.80	926.01	920.91	922.90
HCEE	458.52	923.10	931.10	922.98	925.97
EW	458.80	923.60	931.45	923.80	926.72
EE	461.08	925.81	930.97	926.05	928.04
GE	463.73	931.52	936.71	931.67	933.68
WE	466.10	937.95	945.95	937.92	941.21
Chen	467.10	937.98	943.45	938.35	940.24
MW	469.44	945.04	952.74	945.12	948.03

Discussions and Results

Here, a novel two parameter probability distribution by integrating a power transformation called "The Beta-Power Half-Cauchy Exponential Distribution" were generated. The various basic properties of the model are generated. Model is described using the graphical plots and was validated practically taking fatigue-life data set. The graphical pots of pdfs showed model as unimodal in nature while the hrf plots confirms that the hazard rate was inverted bathtub shaped, increasing and decreasing in nature. The MLE estimates of parameters alpha and beta were 1.5425 and 999.9612 with SE of 0.139 and 90.05 respectively. The LL and the information criterion values are studied and p value of 0.648 under the KS test confirms significant fitting. To validate the model graphically, P-P and Q-Q plots were obtained confirming the model suitability. To check the superiority, other seven probability models accessible in the literature are compared and found to be better fitting as compared to the competing models.

The Bayesian analysis and the other important analysis and testing the model on various data sets will be the further study of the proposed model. Proposed BPHCE model will help in analyzing the data of different nature finding in modern theory as well as in practical life as well as for the research for getting literature related to the probability distribution and its applications.

Limitations: The main limitation is applying it on a single fatigue data and using only one parameter estimation technique MLE. it should be applied on various datasets of different fields.

Data availability: all the data sets used in study and generated during the analysis are included in the study.

Conflict of Interest: Authors have no conflict of interest.

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Authors Contribution: Model formulation, Analysis and manuscript writing (Lal Babu Sah Telee); Interpretations, revisions and conclusion (Kapil Shah and Arun Kumar Chaudhary), Proof reading (Vijay Kumar).

Conclusions

This study presented a flexible two parameter Beta-Power Half -Cauchy Exponential (BPHCE) distribution developed by integrating a power transformation into the half-Cauchy exponential framework. The model has closed form cumulative distribution function and models the positive data with tails and skewness effectively.

MLE was employed suitability is tested using the well-known fatigue-life dataset with goodness-of -fit test and graphical assessments confirming its adequacy. The comparative analysis based on information criteria demonstrated that the BPHCE outperforms several established lifetime models.

The BPHCE distribution thus proves to be a robust and versatile tool for reliability and survival analysis, particularly suited for datasets exhibiting complex tail behavior. Future work may explore multivariate extensions, regression modeling, and applications in other fields such as finance, environmental science, and medical research.

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