Censored Negative Exponential Distribution as a Mixed Distribution and Derivation of Its Moments

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Abstract
Censored negative exponential distribution is treated as a mixed type distribution having two distinct types of components. These components give arise to continuous as well as discrete random variables. Moments (mean and variance) are derived for the doubly censored, right censored, and left censored negative exponential distributions (NEDs) along with separations of continuous and discrete components and their respective means and variances. Moments obtained for the censored NEDs are then compared to the corresponding values of the uncensored NEDs and the changes in the proportions of the moments due to censoring are examined and assessed. Plots of moments of the censored distributions including a three dimensional scatter plot are presented considering different hypothetical values at which censoring may occur. These distributions are widely applied in fitting and modeling failure time data in survival and reliability analyses.

Key words: censoring, exponential distribution, failure time data, mixed distribution, moments

Introduction
Usually probability distributions are either discrete or continuous types. Occasionally, a probability distribution can be a combination of discrete as well as continuous random variables. Such distributions are referred as mixed type distributions. A mixed distribution can be decomposed into two components with one component leading to discrete random variable and another leading to continuous random variable. Alternatively, discrete and continuous distributions can be combined together to form a mixed distribution (Shrestha 2011). Mixed distributions have been used in different areas such as modeling daily amount of rainfall data (Sushaila et al. 2011), insurance claim data in actuarial statistics, etc. In addition, probability distribution of a censored random variable can be treated as the probability distribution of a mixed random variable. Consequently, moments of a censored random variable can be derived using the principle of mixed distribution. The paper specifically deals with censored negative exponential distribution (NED) as a mixed distribution since the distribution is widely used in fitting and modeling failure time data in survival and reliability analyses (Hogg & Tanis 2001). Moments, specifically, mean and variance are derived for doubly and singly censored negative exponential distributions and are compared with the corresponding values of the moments of the distributions without censoring. The true values of the mean and variance of a random variable like failure time are the values obtained for uncensored data. Censoring take place due to some unavoidable compulsions such as fixation of the study period, etc. (Leung et al. 1997). Consequently, computations of mean and variance for censored data are different from the corresponding computations for uncensored distribution. The paper has attempted to derive functional relationships between the parametric values of uncensored and censored data under the assumption of NED of failure time data and examines in what proportions do the values for the moments change due to censoring. Moreover, failure time data can be characterized by different types of censoring. In the paper, censoring only due to the fixation of the study period (lower and / or upper limits of the failure time) is considered and no other types of censoring are assumed to occur during the study period.
Methodology

The principles of mixed distribution are used to associate it with censored distribution as follows (UAH 2012). A mixed type random variable can be associated with the Bernoulli random variable as an indicator variable with parameter \( P_C \) such that \( I = 1 \) in case of success and \( X \) is distributed as a continuous random variable \( X_C \). On the other hand \( I = 0 \) in case of failure then \( X \) is distributed as a discrete random variable \( X_D \). Let \( F(x) \) be the distribution function (DF) of the mixed random variable, \( F_D(x) \) is the DF of the corresponding discrete random variable \( (X_D) \) which is the discrete component of the mixed distribution, and \( F_C(x) \) is the DF of continuous random variable \( (X_C) \) which is the continuous component of the mixed distribution. Let \( D_F \) be a nonempty set of discontinuities such that \( P(X \in D_F) \in (0,1) \). And \( D_F^c \) is the complement of. Thus, and. and are positive constants both adding to unity. Also, and . It follows that

\[
R(X \leq x) = R(X \in D_F^c \cup X \in D_F) = P(X \in D_F^c) + P(X \in D_F)
\]

so that

\[
P(X \leq x) = P(X \in D_F^c) + P(X \leq x | X \in D_F) \cdot P(X \in D_F)
\]

Thus, \( R_C(x) = \frac{P(X \in D_F^c) + P(X \leq x | X \in D_F)}{P(X \in D_F)} \) (1)

Also,

\[
P(X = x) = P(X = x | X \in D_F) \cdot P(X \in D_F)
\]

\[
P(X = x | X \in D_F) = \frac{P(X = x \cap X \in D_F)}{P(X \in D_F)} = \frac{P(X = x)}{P(X \in D_F)}
\]

Thus, \( P(X = x) = \begin{cases} P(X = x) & \text{if } X \in D_F^c \\ \frac{P(X = x)}{P(X \in D_F)} & \text{if } X \in D_F \end{cases} \) (2)

Mean and variance of a mixed random variable can be obtained by adding means and variances from discrete and continuous distributions, respectively (UIOWA 2004).

Results and Discussion

Results are presented in different sections as follows.

Doubly censored negative exponential distribution

For a doubly censored NED, a new variable \( Y \) is defined as follows.

\[
Y = \begin{cases} X & \text{if } a \leq X \leq b \\ a & \text{if } X < a \\ b & \text{if } X > b \end{cases}
\]

where \( a \) and \( b \) are nonnegative constants and \( X \) is distributed as negative exponential distribution with parameter \( \lambda \). The DF of the new variable is:

\[
G(y) = \begin{cases} 0 & \text{if } y < a \\ 1 - e^{-\lambda y} & \text{if } a \leq y < b \\ 1 & \text{if } y \geq b \end{cases}
\]

so that \( P(Y = a) = 1 - e^{-\lambda a} \) and \( P(Y = b) = e^{-\lambda b} \).

Consequently, the distribution is a mixed distribution since \( Y \) is discrete at \( a \) and \( b \) and continuous in the interval \((a, b)\). Graph of the DF is shown below (Fig. 1) for \( \lambda = 1 \), and . The graph depicts continuous curve between \( a \) and \( b \) which is due to continuous nature of the mixed variable in the range \((a, b)\). At \( a \), the distribution function jumps from 0 to \( 1 - e^{-\lambda a} = 0.3935 \) and at \( b \) distribution function jumps from \( 1 - e^{-\lambda b} = 0.8647 \) to 1 which is due to the discrete nature of the variable at \( a \) and \( b \).

Mean and variance of the distribution are derived as follows.

Mean

Mean of the mixed negative exponential distribution can be computed by adding the mean from the discrete values \((a \text{ and } b)\) and mean from the interval \((a, b)\) as follows.

Mean from discrete values is:

\[
\alpha = P(Y = a) \times a + P(Y = b) \times b = \alpha(1 - e^{-\lambda a}) + b e^{-\lambda b}
\]
Mean from interval is:
\[ \int_0^\delta ye^{-\lambda y} dy = \lambda \int_0^\delta ye^{-\lambda y} dy \]
so that the mean of \( Y \) is:
\[ E(Y) = \alpha \cdot F(Y = \alpha) + 1 \cdot \int_0^\delta ye^{-\lambda y} dy \]
\[ = \alpha[1 - e^{-\alpha}] + be^{-b} + \lambda \int_0^\delta ye^{-\lambda y} dy \]
where \( \int_0^\delta ye^{-\lambda y} dy \) is evaluated as follows. Let
\[ u = y \text{ and } v = e^{-\lambda y} \text{ so that} \]
\[ \int_0^\delta ye^{-\lambda y} dy = \int_0^\delta u e^{-v} dv \]
\[ = \left[ -ve^{-v} \right]_0^\delta \text{ evaluated as follows. Let} \]
\[ \alpha = \frac{\delta}{\lambda} \text{ so that} \]
\[ = \frac{1}{\lambda} \left( \alpha e^{\alpha} - e^{\alpha}(\alpha + 1) \right) \]
Thus, mean of the doubly censored NED is
\[ E(Y) = a[1 - e^{-\alpha}] + be^{-b} + \frac{1}{\lambda} \left( \alpha e^{\alpha} - e^{\alpha}(\alpha + 1) \right) \]
Variance
As above, \( E(Y^2) \) is derived as follows.
where \( \int_0^\delta y^2 e^{-\lambda y} dy \) is evaluated as follows. Let
\[ u = y^2 \text{ and so that} \]
\[ E(Y^2) = \alpha^2 \cdot F(Y = \alpha) + 1 \cdot \int_0^\delta y^2 e^{-\lambda y} dy + \lambda \cdot F(Y = \delta) \]
\[ = \alpha^2[1 - e^{-\alpha}] + \delta \int_0^\delta y^2 e^{-\lambda y} dy + \lambda \int_0^\delta y^2 e^{-\lambda y} dy \]
Thus, variance of the doubly censored negative exponential distribution is
\[ \text{Var}(Y) = \frac{1}{\lambda^2} \left( \alpha e^{2\alpha} - 2e^{\alpha}(\alpha + 1) + e^{2\alpha} \right) \]
Right Censored Negative Exponential Distribution
For right censored distribution, a new variable \( Y \) is defined as follows:
\[ Y = \begin{cases} X & \text{if } X \leq b \\ b & \text{if } X > b \end{cases} \]
The CDF of the new variable is:
\[ F(Y = x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } 0 \leq x < b \\ 1 & \text{if } x \geq b \end{cases} \]
\( \lambda \) is discrete at \( \delta \) so that \( F(Y = b) = e^{-\lambda b} \) and continuous in the range \([0, b]\). Graph of the CDF is shown below for \( \lambda = 1 \) and \( b = 2 \). Mean and variance of the right censored distribution are derived as follows.
Mean
\[ E(Y) = \frac{1}{\lambda} \left( a\lambda + e^{-\lambda a} - e^{-\lambda b} \right) \]
is the mean of the doubly censored NED. For \( a = 0 \), mean is

\[ E(Y) = \frac{1}{\lambda} \left( 1 - e^{-\lambda b} \right). \]  

Variance
Variance of the doubly censored NED is
\[ \sigma^2(Y) = \frac{2}{\lambda^2} \left( e^{-\lambda a} + e^{-\lambda b} \right) \left( \frac{2}{\lambda} \left( e^{-\lambda a} + e^{-\lambda b} \right) - \frac{1}{\lambda^2} \left( e^{-2\lambda a} + e^{-2\lambda b} \right) \right) \]
For \( a = 0 \), variance is:
\[ \sigma^2(Y) = \frac{1}{\lambda^2} \left( 1 - e^{-2\lambda b} \right) = \frac{2}{\lambda} e^{-\lambda b} \]

Left censored negative exponential distribution
For left censored distribution, a new variable \( Y \) is defined as follows:
\[ Y = \begin{cases} 
  a & \text{if } X < a \\
  X & \text{if } X \geq a
\end{cases} \quad (11) \]
The CDF of the new variable is:
\[ G(y) = \begin{cases} 
  0 & \text{if } Y < a \\
  1 - e^{-\lambda y} & \text{if } Y \geq a
\end{cases} \quad (12) \]

Mean
\[ E(Y) = \frac{1}{\lambda} \left( a\lambda + e^{-\lambda a} - e^{-\lambda b} \right) \]
is the mean of the doubly censored NED. For \( b = \infty \), mean is

\[ E(Y) = \frac{1}{\lambda} \left( a\lambda + e^{-\lambda a} \right). \]  

Variance
Variance of the doubly censored NED is:
\[ \sigma^2(Y) = \frac{2}{\lambda^2} \left( e^{-\lambda a} - e^{-\lambda b} \right) \left( \frac{2}{\lambda} \left( e^{-\lambda a} - e^{-\lambda b} \right) - \frac{1}{\lambda^2} \left( e^{-2\lambda a} - e^{-2\lambda b} \right) \right) \]
For \( b = \infty \), variance is:
\[ \sigma^2(Y) = \frac{1}{\lambda^2} \left( 1 - e^{-2\lambda a} \right) \]

Decomposition of the doubly censored negative exponential distribution
Mixed distribution of doubly censored NED can be decomposed into discrete and continuous components assuming the two components as events of a Bernoulli trial with success corresponded to continuous random variable and failure corresponded to discrete random variable. If \( P_C \) is the probability of the mixed random variable being continuous type and \( P_D \) is the probability of the mixed random variable being discrete type (Fig. 2) then \( P_C + P_D = 1 \).
Consequently,
Mean of the discrete component is:

$$E(Y_D) = \gamma_1 P(Y_D = \gamma_1) + \gamma_2 P(Y_D = \gamma_2)$$

$$= a \cdot \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}} + b \cdot \frac{e^{-\lambda b}}{1 - e^{-2\lambda a} + e^{-\lambda b}}$$

$$= a \cdot \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}} + b \cdot \frac{e^{-\lambda b}}{1 - e^{-2\lambda a} + e^{-\lambda b}}$$

Similarly,

$$E(Y_D^2) = \gamma_1^2 P(Y_D = \gamma_1) + \gamma_2^2 P(Y_D = \gamma_2)$$

$$= a^2 \cdot \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}} + b^2 \cdot \frac{e^{-\lambda b}}{1 - e^{-2\lambda a} + e^{-\lambda b}}$$

Variance of the discrete component is:

$$Var(Y_D) = E(Y_D^2) - (E(Y_D))^2$$

$$= a^2 \cdot \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}} + b^2 \cdot \frac{e^{-\lambda b}}{1 - e^{-2\lambda a} + e^{-\lambda b}} - \left\{ a \cdot \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}} \right\}^2$$

$$= \frac{(a - b)^2 (1 - e^{-\lambda a}) e^{-\lambda b}}{(1 - e^{-2\lambda a} + e^{-\lambda b})^2}$$

Continuous Component

CDF of continuous component, YC is obtained as follows.

$$G(y) = 1 - e^{-\psi}, G_D(y) = \frac{1 - e^{-\lambda a}}{1 - e^{-2\lambda a} + e^{-\lambda b}}, a \leq y < b$$

$$G(y) = 1; G_D(y) = 1, y \geq b$$

Using $G_C(y) = \frac{G(y) - P_D C_D(y)}{P_C}$,

$$P_C = 1 - e^{-\lambda a} + e^{-\lambda b} P_D = e^{-\lambda a} - e^{-\lambda b},$$

we obtain the CDF of YC as follows.

In the interval, $a \leq y < b$
Thus, the CDF of the continuous component of the mixed distribution can be expressed as:

\[ G_C(y) = \begin{cases} 
0 & \text{if } y < a \\
\frac{e^{-\lambda a} - e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}} & \text{if } a \leq y < b \\
1 & \text{if } y \geq b
\end{cases} \tag{19} \]

\[ g_C(y) = \frac{\partial G_C(y)}{\partial y} = \frac{\lambda e^{-\lambda a} - e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}} \tag{20} \]

Mean of \( Y \) is:

\[ E(Y) = \int_0^b yg_C(y)dy = \frac{2}{\lambda(\lambda a - e^{-\lambda b})} \int_0^b ye^{-\lambda y}dy \]

\[ = \frac{1}{\lambda e^{-\lambda a} - e^{-\lambda b}} \left[ e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}((a\lambda + 1)) \right] \tag{21} \]

Also,

\[ Var(Y) = E(Y^2) - [E(Y)]^2 \]

\[ = \frac{2}{\lambda(\lambda a - e^{-\lambda b})} \left[ \int_0^b ye^{\lambda y}dy + \frac{2}{\lambda} \int_0^b e^{\lambda y}((a\lambda + 1) - e^{\lambda y}(a\lambda + 1)) \right] \]

\[ = \frac{1}{\lambda(\lambda a - e^{-\lambda b})} \left[ \frac{\lambda}{\lambda - e^{-\lambda a}}(a\lambda + 1) - e^{-\lambda a}(a\lambda + 1) \right] \tag{22} \]
Decomposition of the right censored negative exponential distribution

The mixed distribution of right censored NED can be decomposed into discrete and continuous components as follows.

\[ P_D = \mathbb{P}(Y = b) = e^{-\lambda b} \] so that \( P_C = 1 - e^{-\lambda b} \).

**Discrete Component**

PMF of \( Y_D \):

\[ f_D(y) = \begin{cases} 0 & \text{if } y < b \\ 1 & \text{if } y = b \end{cases} \]  

DF of \( Y_D \):

\[ G_D(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 & \text{if } y \geq 0 \end{cases} \]  

Mean of the discrete component is: \( E(Y_D) = b \)  

Variance of the discrete component is: \( \text{Var}(Y_D) = 0 \)

**Continuous Component**

CDF of continuous component, \( Y_C \) is obtained as follows.

\[ G(y) = 1 - e^{-\lambda y}; \quad G_D(y) = 1; \]

Using \( G_D(y) = \frac{G(y) - P_D G_D(y)}{P_C} \), we obtain the CDF of as follows.

In the region, \( y \geq 0 \)

\[ G_C(y) = \frac{G(y) - P_D G_D(y)}{P_C} = \frac{1 - e^{-\lambda y} - (e^{-\lambda b}) \times 1}{1 - e^{-\lambda b}} \]

\[ = 1 - \frac{e^{-\lambda y}}{1 - e^{-\lambda b}} \]

Thus, the CDF of the continuous component of the mixed distribution can be expressed as:

\[ G_C(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \frac{e^{-\lambda y}}{1 - e^{-\lambda b}} & \text{if } 0 \leq y < b \\ 1 & \text{if } y \geq b \end{cases} \]  

PDF of \( Y_C \) is:

\[ g_C(y) = \frac{\partial G_C(y)}{\partial y} = \frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda b}}, \quad 0 \leq y < b \]  

Mean of \( Y_C \) is:

\[ E(Y_C) = \int_0^b y g_C(y) dy = \frac{\lambda}{1 - e^{-\lambda b}} \int_0^b y e^{-\lambda y} dy \]

\[ = \frac{1}{\lambda} \frac{b e^{-\lambda b} - \lambda}{1 - e^{-\lambda b}} \]  

Also,

\[ E(Y_C^2) = \int_0^b y^2 g_C(y) dy = \frac{\lambda}{1 - e^{-\lambda b}} \int_0^b y^2 e^{-\lambda y} dy \]

\[ = \frac{2}{\lambda^2} \left( \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{2 b e^{-\lambda b}}{\lambda} + \frac{1}{\lambda^2} \right) \]  

\[ \text{Var}(Y_C) = E(Y_C^2) - [E(Y_C)]^2 \]

\[ = \frac{2}{\lambda^2} \left( \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{2 b e^{-\lambda b}}{\lambda} + \frac{1}{\lambda^2} \right) \]

\[ \text{Var}(Y_C) = \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{2 b e^{-\lambda b}}{\lambda(1 - e^{-\lambda b})} \]  

It can be verified that combination of the discrete and continuous components give arise to the mixed distribution as follows.

\[ E(Y) = P_D E(Y_D) + P_C E(Y_C) \]

\[ = (e^{-\lambda b}) \times b + \left( 1 - e^{-\lambda b} \right) \left( \frac{b^2 e^{-\lambda b}}{\lambda^2} \right) \]

\[ = \frac{1}{\lambda} (1 - e^{-\lambda b}) \]

The expression is identical with the mean of the mixed random variable as given in Equation 9.

Also,

\[ \text{Var}(Y) = P_D \text{Var}(Y_D) + P_C \text{Var}(Y_C) + P_D P_C (\text{Var}(Y_D) + \text{Var}(Y_C))^2 \]
The expression is identical with the variance of the mixed random variable as given in Equation 10.

**Comparative assessment of the moments between censored and uncensored NEDs**

The tables below display values of the means and variances of the uncensored and censored NEDs, proportions of right censored NED means and variances compared to uncensored NED means and variances, respectively (for $\lambda = 0.2$). Tables demonstrate that as the time of right censoring increases the proportions of the means and variances of the right censored NED compared to uncensored NED also increases and approaches to unity for censoring times equivalent 40 years and 50 years, respectively if mean failure time is 5 years for uncensored NED (Table 2 & 3). The Tables also indicate that if the time of censoring is around the mean failure time then the mean and variance obtained from right censored distribution are much lower than the actual mean and variance of the uncensored NED. The values obtained in the tables can be utilized when estimates are obtained from right censored NED and we are interested to make assessment on the actual parameter values for the uncensored NED. The actual estimates for uncensored NED are likely to the values obtained by dividing the estimates obtained from right censored NED by the proportions shown in the Tables for different times at censoring.

### Table 1. Moments of negative exponential distribution

<table>
<thead>
<tr>
<th>Distribution Type</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncensored</td>
<td>$\frac{1}{\lambda}$</td>
<td>$\frac{1}{\lambda^2}$</td>
</tr>
<tr>
<td>Doubly Censored</td>
<td>$\frac{1}{\lambda} \left(a+e^{-\lambda b} - e^{-\lambda b}\right)$</td>
<td>$\frac{2}{\lambda^2} \left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda} \left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{1}{\lambda^2} \left(e^{-2\lambda a} - 2e^{-\lambda (a+b)} + e^{-2\lambda b}\right)$</td>
</tr>
<tr>
<td>Right Censored</td>
<td>$\frac{1}{\lambda} \left(1 - e^{-\lambda b}\right)$</td>
<td>$\frac{1}{\lambda^2} \left(1 - e^{-2\lambda b}\right) - \frac{2}{\lambda} e^{-\lambda b}$</td>
</tr>
<tr>
<td>Left Censored</td>
<td>$\frac{1}{\lambda} \left(a+e^{-\lambda a}\right)$</td>
<td>$\frac{1}{\lambda^2} \left(2e^{-\lambda a} - e^{-2\lambda a}\right)$</td>
</tr>
</tbody>
</table>

**Three dimensional scatter plot of mean in doubly censored NED**

A typical three dimensional scatter plot of mean for different hypothetical values at which censoring can take place is plotted for an assumed value of mean in uncensored distribution. In the plot, $a$ and $b$ values are assigned in $X$ and $Y$ planes and means are assigned in the $Z$ plane. The plot shows increasing values of the mean with increasing values of $a$ and $b$. 

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Censored NED is treated as the mixed type probability distribution having two components with one component leading to a discrete random variable and another leading to a continuous random variable. Consequently, moments of the censored NED are derived using the principle of mixed distribution. Since NED is widely used as a parametric distribution in modeling varied types of data such as failure time data in survival and reliability analysis (Hosmer et. al. 2008), insurance claim data in actuarial statistics, warranty claim data in consumer product sales, etc., it is
considered in this paper under the situations of censoring. Moments, specifically, mean and variance are derived for doubly and singly NEDs and are compared with the corresponding values of the moments of the NED without censoring. The values of the mean and variance of failure time are often observed along with censoring which leads inaccurate estimation of the true values of the moments if estimates applicable for only uncensored data are used. Thus, in the paper, functional relationships between the parametric values of uncensored and censored NEDs of failure time are derived and changes in proportions of the moments due to censoring are examined and assessed. For instance, as an illustration, the functional relationship shows that mean of failure time for $\lambda = 0.2$ is significantly less (proportion = 0.8647) than the true mean of uncensored NED if time at right censoring is 10 years and approaches to true mean if time at right censoring is about 30 years. Consequently, sample estimate of mean obtained from censored data is also likely to be less by a similar proportion as found for its parametric value. Similarly, the functional relationship shows that variance of failure time for $\lambda = 0.2$ is significantly less (proportion = 0.8531) than the true variance of uncensored NED if time at right censoring is 20 years and approaches to true variance if time at right censoring is about 50 years. Consequently, sample estimate the variance obtained from censored data is also likely to be less by a similar proportion as found for its parametric value. Moreover, failure time data can be characterized by different types of censoring. In the paper, censoring only due to the fixation of the study period (lower and upper limits of the failure time) is considered and no censoring occurs in between the study period.

References


