Nepal Journal of Science and Technology 10 (2009) 155-159

Low-dimensional Nilpotent Lie Groups G_4

Chet Raj Bhatta
Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu
e-mail: crbhatta @yahoo.com

Abstract

An uncertainty principle due to Hardy for Fourier transform pairs on R says that if the function f is "very rapidly decreasing" then the Fourier transform cannot also be "very rapidly decreasing unless f is indentically zero." In this paper we study the relevant data for G_4 and state and prove an analogue of Hardy theorem for low-dimensional nilpotent Lie groups G_4 .

Key words: Fourier transform, uncertainity principle, Nilpotent Lie groups

Introduction

Uncertainty principles related to decay of function and its Fourier transform have a long history, starting from a theorem due to Hardy (see Hardy. 1933 H. Rieter and J.D. Stegman (2000)). Hardy's theorem consists of two parts: in the first it characterises the heat kernel interms of its decay and that of its Fourier transform and in the second it shows that a non-zero function and its Fourier transform can have no faster decay. Several generalisations of the second part of the Hardy theorem has appeared since, most notable among them being the result of Cowling & Price (1983), Benedetto (1975)).The theorem of Cowling & Price says roughly that it is not possible for a nonzero function and its Fourier transform both to decay very rapidly. To be precise, let $e_a(x) = \exp(ax^2)$; then if $\|e_a f\|_p < \infty$, $\|e_b \hat{f}\|_q < \infty$, min $(p.q.) < \infty$ and ab^3 $\frac{1}{4}$, $\geq \frac{1}{4}$ then $f = 0$. The case $p = q = \infty$ was treated by Hardy.

Recently considerable attention has been paid to proving analogue of Hardy's theorem and its L^p version in the setup of non commutative groups (Astengo *et al.* (2002), Bagchi & Ray (1998), Cowling M.G. *et al.* (2000), Eguchi, *et al.* (2000), Folland & Sitaram, A. (1997), Ole (1983), & Sitaram, Sundari (1997) generalised the second part of Hardy's theorem to connected semi-simple Lie groups with one congugacy class of Carten subgroups and to the K-invariant case for general connected semisimple Lie groups. The result was extended to all semisimple Lie groups with finite Centre by Cowling, *et al.* (2000) . Kumar & Bhatta (2004) have proved the analogue of Cowling and Price theorem for $G_n = G_{3,1}, G_{5,1}, G_{5,3}$ or $G_{5,6}$ and $ab > 0$.

In this paper our aim is study the corresponding connected and simply-connected Lie groups and its Coadjoint orbits, irreducible unitary representation and related data for G_4 as well as Parseval identity. Also we prove the analogue of Hardy's theorem for low-dimensional nilpotent Lie groups G_4.

Calculation of relavant data for G_4

$g = g_4 = \hat{A}X_1 + \ldots + \hat{A}X_4$

The data computed for G_4 have been presented in eight sections.

(I) $g = g_4 = \Re X_1 + \ldots + \Re X_4$

$[X_4, X_3] = X_2 \qquad [X_4, X_2] = X_1$

(I) $G = G_4 = R^4$

$$(x_1,\ldots,x_4)(y_1,\ldots,y_4) = (x_1 + y_1 + x_4 y_2 + \tfrac{1}{2}x_4^2 y_3, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4)$$

$$(x_1,\ldots,x_4)^{-1} = (-x_1 + x_2 x_4 - \tfrac{1}{2}x_3 x_4^2, -x_2 + x_3 x_4, -x_3, -x_4)$$

(I I) $X_1 = \frac{\partial}{\partial x_1}$

$$X_2 = x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

$$X_3 = \frac{1}{2} x_4^2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

$$X_4 = \frac{\partial}{\partial x_4}$$

(I I I) $\exp(a_1 x_1 + \ldots + a_4 x_4) = (a_1 + \tfrac{1}{2}a_2 a_4 + \tfrac{1}{6}a_3 a_4^2, a_1 + \tfrac{1}{2}a_3 a_4, a_3, a_4)$

$$\log(x_1,\ldots,x_4) = (x_1 - \tfrac{1}{2}x_2 x_4 + \tfrac{1}{12}x_3 x_4^2)X_1 + $$
$$+ (x_2 - \tfrac{1}{2}x_3 x_4)X_2 + x_3 X_3 + x_4 X_4$$

(I V) $\mathrm{Ad}(x_1,\ldots,x_4)(a_1 X_1,\ldots, + a_4 X_4)$

$$= (a_1 + x_4 a_2 - x_2 a_4 + \tfrac{1}{2}x_4^2 a_3)X_1 + (a_2 + x_4 a_3 - x_3 a_4)X_2 + a_3 X_3 + a_4 X_4$$

$$Ad^*(x_1,\ldots,x_4)(\xi_1 f_1,\ldots,\xi_4 f_4)$$

$$= \xi_1 f_1 + (\xi_2 - x_4\xi_1) f_2 + (\xi_3 - x_4\xi_2 + \frac{1}{2}x_4^2\xi_1) f_3 + (\xi_4 + x_3\xi_2 + (x_2 - x_3 x_4)\xi_1) f_4$$

(V) $f = \xi_1 f_1 + \ldots + \xi_4 f_4$

(a) $\xi_1 \neq 0$

$$\vartheta(f) = \Re X_1 + \Re(\xi_2 X_2 - \xi_1 X_2)$$

$d = 2$; $j_1 = 2$ $j_2 = 4$

$$P_3(t_1, t_2) = \frac{1}{2\xi_1} t_1^2 + \left(\xi_2 - \frac{\xi_2^2}{2\xi_1} \right)$$

(b) $\xi_1 = 0, \xi_2 \neq 0$:

$$\vartheta(f) = \Re X_1 + \Re X_2$$

$d = 2\ ;\ j_1 = 3,\ \ j_2 = 4$

(c) $\xi_1 = \xi_2 = 0\ x_1 = x_2 = 0$

$\vartheta(f) = \vartheta$

(VI) $f = \xi_1 f_1 + \ldots + \xi_4 f_4$

(a) $\xi_1 \neq 0,\ \ \xi_2 = \xi_4 = 0$

π_f can be realized on $L^2(\Re)$ by

$[\pi_f(x_1, \ldots, x_4)\phi](t)$

$= \exp\ \xi\xi\ \phi(t - x_4)$

(b) $\xi_1 = 0, \xi_2 \neq 0, \xi_3 = \xi_4 = 0$

π_f can be realized on $L^2\ (\Re)$ by

$[\pi_f\ (x_1, \ldots, x_4)\ \phi]\ (t)$

$= \exp\,[2\pi i\,(x_2 - x_3 t)\,\xi_2]\,\phi\,(t - x_4)$

(c) $\xi_1 = \xi_2 = 0$

π_f is the character given by

$\pi_f\,(x_1, \ldots, x_4) = \exp\,[2\pi i\,(x_3\,\xi_3 + x_4\,\xi_4)]$

(VIII) $W = \{(\xi_1, \xi_3) \in \Re^2 : \xi_1 \neq 0\}$

$\theta(0, \ldots, 0) = \int_w \operatorname{tr}\pi_{\xi_1 f_1 + \xi_3 f_3}\,(\theta)\,|\xi_1|\,d\lambda_2\,(\xi_1, \xi_3)$

Parseval identity for G_4

$(x_1, x_2, x_3, x_4)^{-1} = (-x_1 + x_2 x_4 - \tfrac{1}{2} x_3 x_4^2, -x_2 + x_3 x_4, -x_3, -x_4)$

For $\xi \neq 0$

$\text{eq}\ \hat{}f\,(\pi_{\xi_1, \xi_3}) = \int_{\Re^4} f(x)\,\pi_{\xi_1, \xi_3}\,(x^{-1})\,dx$

For $\phi \in L^2\ (\Re)$, we have

$(\text{eq}\ \hat{}f\,(\pi_{\xi_1, \xi_3})\phi)\,(y) = \int_{\Re^4} f(x)\,\pi_{\xi_1, \xi_3}\,(-x_1 + x_2 x_4 - \text{eq 12}\ x_3 x_4^2, -x_2 + x_3 x_4, -x_3, -x_4)\,\phi(y)\,dx$

$= \int_{\Re^4} f(x)\,\exp 2\pi i\,(-x_1 + x_2 x_4 - \text{eq 12}\ x_3 x_4^2 - (-x_2 + x_3 x_4)y - \text{eq 12}\ x_3 y^2)\,\xi_1 - x_3\xi_3)]\,\phi(y + x_4)\,dx$

Applying $x_4 \to x_4 - y$

$X = \exp\,(2\pi i\,[-x_1 + x_2(x_4 - y) - \text{eq 12}\ x_3(x_4 - y)^2 - (-x_2 + x_3)\,(x_4 - y)y - \text{eq 12}\ x_3 y^2)\,\xi_1 - x_3\xi_3)])$

$= \exp\,(2\pi i\,[-x_1 + x_2 x_4 - x_2 y - \text{eq 12}\ x_3 x_4^2) - \text{eq 12}\ x_3 y^2 + x_3 y x_4 + x_2 y - x_3 x_4 y + x_3 y^2 - \text{eq 12}\ x_3 y^2)\,\xi_1 - x_3\xi_3])$

$= \exp\,(2\pi i\,[(-x_1 + x_2 x_4 - \text{eq 12}\ x_3 x_4^2)\,\xi_1 - x_3\xi_3])$

$\therefore$ $(\text{eq}\ \hat{}f\,(\pi_{\xi_1, \xi_3})\phi)\,(y) = \int_{\Re^4} f(x_1, x_2, x_3, x_4 - y)\,\exp\,(-2\pi i)\,[(x_1 - x_2 x_4 + \text{eq 12}\ x_3 x_4^2)\,\xi_1 + x_3\xi_3]\,\phi\,(x_4)\,dx$

$\text{eq}\ \hat{}f\,(\pi_{\xi_1, \xi_3})$ is an integral operator on $L^2(\Re)$ whose Kernel is

$K^f_{(\xi_1, \xi_3)}\,(y, x_4) = \int\!\!\iiint f(x_1, x_2, x_3, x_4 - y)\,\exp\,(-2\pi i\,[(x_1 - x_2 x_4 + \text{eq 12}\ x_3 x_4^2)\,\xi_1 + x_3\xi_3])\,dx_1\,dx_2\,dx_3$

$= F_1 F_2 F_3 f(\xi_1, -x_4\xi_1, \text{eq 12}\ x_4^2\,\xi_1 + \xi_3, x_4 - y)$

$\|\text{eq}\ \hat{}f(\pi_{\xi_1, \xi_3})\|^2 = \int_{\Re^2} |K^f_{(\xi_1, \xi_3)}\,(y, x_4)|^2\,dy\,dx_4$

$= \int_{\Re^2}$ $|F_1 F_2 F_3 f(\xi_1, x_4\xi_1,$ eq 12 $x_4^2 \xi_1 + \xi_3, x_4 - y)|^2 \, dydx_4$

$v \to x_4 - y$

$= \int_{\Re^2}$ $|F_1 F_2 F_3 f(\xi_1, -(v+y)\xi_1,$ eq 12 $(y+v)^2 \xi_1 + \xi_3, v)|^2 \, dy\, dv$

$y \to y - v$

$= \int_{\Re^2}$ $|F_1 F_2 F_3 f(\xi_1, -y\xi_1,$ eq 12 $y^2 \xi_1 + \xi_3, v)|^2 \, dy\, dv$

$y \to$ eq u1

$= |\xi_1|^{-1} \int_{\Re^2}$ ξeq 12$\xi\xi$dudv

(A) $= |\xi_1|^{-1} \int_{\Re^2}$ eq F1 F2 F3 f 1, w, –eq 12 \f($w^2, \xi_1) + \xi_3, v)^2$) dwdv

which is a Parseval identity obtained for G_4

4. Analogue of Hardy theorem:

Theorem 4.1: *Let* $f \in L^1(G_4) \cap L^2(G_4)$ and suppose that

(i) $|f(x_1, x_2, x_3, x_4)| \leq C \exp(-a\pi \|(x_1, x_2, x_3, x_4)\|^2)$

(ii) $\int_{\Re^2}$ $\exp(2b\pi\, \xi_1^{\,2})\|$ eq ^f$(\pi_{\xi_1, \xi_3})\|^2_{HS} |\xi_1| \, d\xi_1 d\xi_3 < \infty$

then $f = 0$ *a.e* *where* $ab > 1$

Proof: Define

$$h(x_1,x_3) = \grave{O}_{i^2} f_{(x_2,x_4)} * f^*_{(x_2,x_4)}(x_1,x_3)\, dx_2 dx_4 \int_{\Re^2} \quad *$$

where, $f_{(x_2,x_4)}(x_1,x_3) = f(x_1,x_2,x_3,x_4)$

$$f^*_{(x_2,x_4)}(x_1,x_3) = \overline{f(-x_1,x_2,-x_3,x_4)}$$

$$|h(x_1,x_3)| \; £ \; \grave{O}_{i^2} \grave{O}_{i^2} |f(y_1,x_2,y_3,x_4)| |(f(y_1 - x_1,x_2,y_3 - x_4,x_4))| \leq \int_{\Re^2}$$

$\leq \exp(-a\pi[\|(y_1, x_2, y_3, x_4)\|^2 + \|(y_1 - x_1, x_2, y_3 - x_3, x_4)\|^2)$

$dx_2 dx_4 dy_1 dy_3$

$\leq \exp([2\|x_2\|^2 + 2\|x_4\|^2 + \|y_1\|^2 + \|y_1 - x_1\|^2 + \|y_3\|^2 + \|y_3 - x_3\|^2])$

$\leq C \exp\; -a\pi 2\; \|2y1 - x1\|2 + \|x1\|2 + \|2y3 - x3\|2 + \|x3\|2$

$\leq C\; \exp -a\pi 2\; [\|x_1\|^2 + \|x_3\|^2]\; \exp -a\pi[\|y_1\|^2 + \|y_3\|^2]\; dy_1 dy_3$

$\leq$ constant $\exp$ **[– a**$\pi 2\; \|x2\|2 + \|x3\|2$ (using invariance)

Now

eq ^h $(\xi_1, \xi_3) = \int_{\Re^2} |F_{1,3}f(\xi_1, x_2, \xi_3, x_4)|^2 dx_2\, dx_4$

$\int_{\Re^2} \exp(2b\,\pi\xi_1^2)$ |eq ^h$(\xi_1, \xi_3)|\, d\xi_1\, d\xi_3$

$= \int_{\Re^2} \exp(2b\pi\,\xi_1^2) \int_{\Re^2} |F_{1,3}f(\xi_1, x_2, \xi_3, x_4)|^2 dx_2\, dx_4\, d\xi_1\, d\xi_3$

$\xi_3 \to \xi_3$ + eq 12 eq x221

$= \int_{\Re^2} \exp(2b\pi\,\xi_1^2) \int_{\Re^2} F_{1,3}f(\xi_1, x_2, \xi_3$ + eq 12 eq x221, $x_4)|^2 dx_2\, dx_4\, d\xi_1\, d\xi_3$ Using (A)

$= \int_{\Re^2} \exp(2b\pi\,\xi_1^2)\, |\xi_1|$ ||eq ^f $(\pi_{\xi_1,\xi_3})\|^2_{HS}\, d\xi_1\, d\xi_3 < \infty$

Hence h = 0 a.e., since eq a2 2b > 1 provided the following holds

Remark 4.2 Let $g \in L^i \cap L^2(\Re)$ such that $|g(x_1, x_2)| \leq C \exp - a\pi(x_1^2 + x_2^2)$

$\int \exp \pi\, by_1^2$ |eq ^g $(y_1, y_2)\,|\, dy_1\, dy_2 < \infty$

g = 0 a.e., where ab > 1

Proof: Fix y_2 = eq ^ and define

$g_{y_2}(x_1) = \int_{\Re} g(x_1, x_2)\; g_{y_2}(x_1) = \grave{O}_i\, g(x_1,x_2)\,\overline{y_2(x_2)}\,dx_2$

$|g_{y_2}(x_1)| \leq \int_{\Re} |g(x_1, x_2)\, dx_2$

$\leq C \int_{\Re} \exp[a\pi(\|x_1\|^2 + \|x_2\|^2)]\, dx_2$

$\leq$ constant $\exp(-a\pi\,\|x_1\|^2)$

Also, $\hat{g}_{y_2}(y_1) = \hat{g}(y_1, y_2)$

$\int_{\Re} \exp(b\pi y_1^2)$ |eq ^$g_{y_2}(y_1)\, dy_1 = \int \exp(b\pi y_1^2)|$ eq ^$g(y_1, y_2)|\, dy_1 < \infty$ for almost all y_2.

So, $g_{y_2} = 0$ almost all y_2, by Cowling-Price theorem $\Rightarrow$Þ $g = 0\,a.e$

References

Astengo, F., Cowling, M.G., Di Blasio B., Sundari M., 2000, Hardy's uncertainty principle on some Lie groups, *J. London Math. Society* **62** (2): 461-472.

Bagchi, S.C. Ray, S. 1998, Uncertainty principle, like Hardy's theorem on some Lie group, *J. Austral. Math. Soc.* (Series A) **65:** 289-302.

Benedetto, J.J. 1975. *spectral analysis.* B.G. Teubner sttutgart

Cowling M.G., Price J.F. 1983. *Generalizations of Heisenberg inequality, Harmonic analysis* (Eds. G. Mauceri, F.Ricci, G. Weiss), LNM, No. 992, Springer Verlin

Cowling, M.G., Sitaram, A., Sundari, M. 2000, Hardy uncertainty principle on semisimple Lie groups, *Pacific J. Math.* **192** : 293-296.

Dym H. and MC. Kean H.P., Fourier 1972, *Fourier Series and integrals*. Academic Press. New York,.

Eguchi, M., Koizumi, S., Kumahara, K. 2000. L^p *Version of the Hardy theorem for motion groups, Journal of Austral. Math. Soc.* (Series A) **68**: 55-67.

Folland G.B., Sitaram, A., 1997, *The uncertainty Principle: A mathematical survey, J. Fourier analysis and application* **3** : 207-238.

G.H. Hardy. 1933.*A theorem concerning Fourier transform. J. London Math. Soc*. **8**:227-231.

H.Rieter and J.D. Stegma. 2000. *Classical Harmonic Analysis and Locally Compact groups*, Oxford Univeristy Press,.

Hewitt, E., and K.A. Ross. 1963 & 1970. *Abstract Harmonic analysis I & II*, Springer Verlag,.

Hörmander L.A. 1991. Uniqueness theorem of Beurling för Fourier transform pairs, *Arkiv för Matematik* **29**(2): 237-240.

Kaniuth, E., Kumar, A.2000. *Hardy's theorem for simply connected nilpotent Lie groups*, Preprint,.

Kumar A., and Chet Raj Bhatta. 2004. *An uncertainty principle like Hardy's theorem for Nilpotent Lie groups*, *J. Aust. Math. Soc.* **76**: 1-7.

Ole, A. Nelsen.1983. Unitary representations and coadjoint orbits of low-dimensional nilpotent Liegroups, *Queens papers in pure and applied mathematics* No. 63, 1983

Sitaram, A., Sundari, M. 1997. An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups. *Pacific J. Math.* **177**.