COWLING PRICE THEOREM FOR LOW DIMENSIONAL NILPOTENT LIE GROUPS

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ABSTRACT

We extend an uncertainty principle due to Cowling and Price to low dimensional Nilpotent Lie groups G_4. The uncertainty principle is a generalization of a classical result due to Hardy.

INTRODUCTION

In the vast literature on uncertainty principles in Harmonic analysis (see [8]), the central theme is the impossibility of simultaneous smallness of a non zero function f and its Fourier transform $\hat{f}$, where $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{IR} f(x) \exp(-2\pi ixy)dx$$

A large number of results, beginning with classical theorem of Hardy (Theorem 1 below), show such impossibility when smallness is interpreted as sharp decay.

In this paper we concern ourselves with results of this kind on Nilpotent Lie groups. We begin by stating the main result of this genre for the real line.

Theorem (Hardy): Let f: R $\rightarrow$ C be measurable and for all x, y

(i) $|f(x)| \leq C\exp(-a\pi x^2)$

(ii) $|\hat{f}(y)| \leq C\exp(-b\pi y^2)$

Where C, a, b > o. If ab > 1 then f = o a. e. If ab = 1 then $f(x) = C\exp(-a\pi x^2)$. If ab < 1 then there exists infinitely many linearly independent functions satisfying (i) and (ii). (See Bagchi, S.C. and et al. 1998, Hardy G.H. 1933, Cowling, M.G. and et al. (2000). Further see Kumar, A and et al., 2004, Sitaram, A and et al. (1997).

Theorem (Cowling-Price): If f: R $\rightarrow$ C be measurable and

(i) $\|e_a f\|_{L^p(R)} < \infty$

(ii) $\|e_b \hat{f}\|_{L^q(R)} < \infty$

Where a, b > o, $e_k(x) = \exp(k\pi x^2)$ and min (p, q) < ∞. If ab $\geq$ 1 then f = o a.e. If ab < 1 then there exist infinitely many linearly independent functions satisfying (i) and (ii).

Theorem (Beurling): for $f \in L^1(R)$, $\int_R\int_R |f(x)|\,|\hat{f}(y)| \exp(2\pi|xy|)\, dxdy < \infty$

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implies f = 0 a.e

For the proof of the above theorems see [9, 3, 12].

Barring the case ab = 1 it is clear that the theorem of Cowling and Price implies the theorem of Hardy. Also the theorem of Beurling implies that of Cowling and Price for ab > 1.

In this paper our aim is to prove Cowling and Price theorem for Nilpotent Lie group G_4 under different conditions. (See the details in Ole 1983).

MAIN RESULTS

Theorem 4: Suppose that f extends analytically to $R\times C\times R^{n-2}$ satisfying

$$|f(x_i, x_2 + iy_2, x_3, ..., x_n)| \le C \exp - a\pi (x_1^2 + \mathrm{Re}\,(x_2 + iy_2)^2 + ... + x_n^2)$$

for some C > o and all $x \in R\times C\times R^{n-2}$ then the function $h(\xi) = ||\xi_1||^{1/2} ||\pi_\xi(f)||_{HS}$ is bounded.

Proof: $|\xi_1| \left\|\pi_\xi(f)\right\|_{HS}^2 = \int_{R^2} |F_{1, ..., (n-1)} f (\xi_1, t, q_\xi (\xi, t) ..., q_{n-1}(\xi, t),s|^2 dtds$

$|F_{1, ..., n-1} f (\xi_1, t, q_\xi (\xi, t), ..., q_{n-1} (\xi, t), s)|$

$= |\int_{R^{n-1}} f(x_1, x_2, ..., x_{n-1}, s) \exp (2\pi i\xi_1 x_1 + 2\pi i t x_2 + ... + 2\pi i q_{n-1} (\xi, t)\, x_{n-1})$

$dx_1 ... dx_{n-1}|$

$x_2 \to x_2 + iy_2$

$= |\int_{R^{n-1}} f(x_1, x_2 + iy_2, ..., x_{n-1}, s) \exp (2\pi i\xi_1 x_1 + 2\pi i t (x_2 + iy_2) + ... + 2\pi i q_{n-1} (\xi, t) x_{n-1})\, dx_1 ... dx_{n-1}|$

$\le \int_{R^{n-1}} | f (x_1, x_2 + iy_2, ..., x_{n-1}, s)| \exp (-2\pi t y_2)\, dx_1 ... dx_{n-1}$

$\le \int_{R^{n-1}} \exp (-a\pi (x_1^2 + x_2^2 + ... + x_{n-1}^2 + s^2) \exp (-2\pi t_2)\, dx_1, ..., dx_{n-1}$

$= \exp (a\pi y_2^2 - 2\pi t y_2 - a\pi s^2) \int_{R^{n-1}} \exp(-a\pi (x_1^2 + x_2^2 + ... + x_{n-1}^2) dx_1, ..., dx_{n-1}$

$\le C \exp (-a\pi s^2) \exp (-2\pi (t y_2 - \frac{a}{2} y_2^2)$

Taking infimum over y_2, we have

$|f_{1, ..., n-1} f (\xi_1, t, q_3 (\xi, t), ... q_{n-1} (\xi, t), s|$

$\le C \exp (-a\pi s^2) \exp (-\pi x^2/_a)$

Thus $h(\xi)$ is bounded.

Theorem: Let $f \in L^1 (G_4) \cap L^2(G_4)$, for $1 \le q < 2$, let us suppose that $ab \ge 2$ with the estimates

(i) $| f (x_1, x_2, x_3, x_4)| \le C \exp (-a\pi ||(x_1, x_2, x_3, x_4)||^2$

(ii) $\int_{R^2} \exp (qb\pi \xi_1^2 || \hat{f} (\pi_{\xi 1, \xi 3})||^q |\xi_1|\, d\xi_1 d\xi_3 < \infty$

then f = 0 a.e

Proof: Let h be the function as above, so that

$|h(x_1, x_2)| \leq C \exp\left(\frac{-a}{2}\pi (\|x_1\|^2 + \|x_3\|^2)\right)$

Let p be such that $1/p + 1/q = 1$

$$\int_{R^2} \exp(b\pi\xi_1^2)|\hat{h}(\xi_1, \xi_3)|\, d_{\xi_1} d_{\xi_3} = \int_{R^2} \exp(b\pi\xi_1^2)|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^2 d\xi_1 d\xi_3 \quad \ldots (1)$$

We want to apply Holders inequality for

$u(\xi_1, \xi_3) = \exp(b\pi\xi_1^2)|\xi_1|^{1/q}\|\hat{f}(\pi_{\xi_1,\xi_3})\|$

and $v(\xi_1, \xi_3) = |\xi_1|^{1/p}\|\hat{f}(\pi_{\xi_1,\xi_3})\|$

$$\int |u(\xi_1, \xi_3)|^q d\xi_1 d\xi_3 = \int \exp(bq\pi\xi_1^2)|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^q d\xi_1 d\xi_3 < \infty$$

$$\begin{aligned}\int |v(\xi_1, \xi_3)|^p d\xi_1 d\xi_3 &= \int |\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^p d\xi_1 d\xi_3\\ &= \int |\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^2 \|\hat{f}(\pi_{\xi_1,\xi_3})\|^{p-1} d\xi_1 d\xi_3\\ &\leq \|f\|_1^{p-2}\int |\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^2 d\xi_1 d\xi_3\\ &= \|f\|_1^{p-2}\|f\|_2\end{aligned}$$

Thus (1) $\Rightarrow \int_{R^2} \exp(b\pi\xi_1^2)|\hat{h}(\xi_1\xi_3)\, d_{\xi_1} d_{\xi_3}$

$\leq \left(\int_{R^2} \exp(b\pi\xi_1^2)|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^q d\xi_1 d\xi_3\right)^{1/q} \times \|f\|_1^{1-2/p}\|f\|_1 < \infty$

So h = 0 a.e. Hence f = 0 a.e

Theorem: For $q \geq 1$, let $f \in L^1(G_4) \cap L^2(G_4)$ satisfying

(i) $|f(x_1, x_2, x_3, x_4)| \leq C \exp(-a\pi\|x\|^2)$

(ii) $\int \exp(qb\pi(\xi_1^2 + \xi_3^2))|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^q d\xi_1 d\xi_3 < \infty$

If ab > 2 then f = 0 a.e

Proof: Let h be as earlier so that

$|h(x_1, x_3)| \leq C \exp\left(\frac{-a}{2}\pi(|x_1|^2 + |x_3|^2)\right)$

Let $\in > o$ and $b' = b - \in$ be such that $ab' > 2$

$\int_{R^2} \exp(b\pi\xi_1^2)\int_{R^2} \exp(b'\pi\xi_1^2)|\hat{h}(\xi_1, \xi_3)| d\xi_1 d\xi_3$

$= \int_{R^2} \exp(b^1\pi\xi_1^2)|\xi_1|\,\|\hat{f}(\pi_{\xi1,\xi3})\| d\xi_1 d\xi_3$

$\leq \int_{R^2} \exp(bq\pi(\xi_1^2 + \xi_3^2))|\xi_1|^{1/q}\|\hat{f}(\pi_{\xi1,\xi3})\|$

$\exp(-\in\pi(\xi_1^2 + \xi_3^2)|\xi_1|^{1/p}\|\hat{f}(\pi_{\xi_1,\xi_3})\| d\xi_1 d\xi_3$

$\leq \int_{R^2} \exp(-qb\pi(\xi_1^2 + \xi_3^2)|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^p d\xi_1 d\xi_3)^{1/p}$

But $\int_{R^2} \exp(-p\in\pi(\xi_1^2 + \xi_3^2)|\xi_1|\,\|\hat{f}(\pi_{\xi_1,\xi_3})\|^p d\xi_1 d\xi_3$

$\leq \|f\|_1^p \int_{R^2} \exp(-p \in \pi(\xi_1^2 + \xi_3^2))\ |\xi_1|\ d\xi_1 d\xi_3$

$\leq k\ \|f\|_1^p$

Hence $\int_{R^2}\ \exp(b'\pi\,\xi_1^2)\ |\,\hat{h}\,(\xi_1, \xi_2)|\ d\xi_1 d\xi_3$

$\leq k\ \|f\|_1\ (\int_{R^2} \exp(qb\pi(\xi_1^2 + \xi_3^2))\ |\xi_1|\ \ \|\,\hat{f}\ (\pi_{\xi_1, \xi_3})\|^q)\ d\xi_1 d\xi_3 < \infty$

So h = 0 a.e and thus f = 0 a.e

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