SOME DISCRETE OPTIMIZATION PROBLEMS WITH HAMMING AND H-COMPARABILITY GRAPHS

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ABSTRACT

Any H-comparability graph contains a Hamming graph as spanning subgraph. An acyclic orientation of an H-comparability graph contains an acyclic orientation of the spanning Hamming graph, called sequence graph in the classical open-shop scheduling problem. We formulate different discrete optimization problems on the Hamming graphs and on H-comparability graphs and consider their complexity and relationship. Moreover, we explore the structures of these graphs in the class of irreducible sequences for the open shop problem in this paper.

INTRODUCTION

We consider a strongly NP-hard open-shop scheduling problem $O \mid\mid C_{max}$, where each job $i \in \{1, \dots, n\}$ has to be processed on each machine $j \in \{1, 2, \dots, m\}$ exactly once without preemption for the positive time p_{ij}. Assume that each machine can process at most one job at a time and each job can be processed on at most on machine at a time. Let $P = [p_{ij}]$, $SIJ = \{o_{ij} \mid p_{ij} > 0\}$ and $C = [C_1, \dots, C_n]$ be the matrix of processing times, the set of all operations and the vector of completion times of all jobs, respectively, so that $C_{\max} = \max_{i \in I} C_i$ and $C_{\max} = \max_{ij} c_{ij}$ hold. A sequence is represented either by an acyclic digraph (sequence graph) $G = (SIJ, E)$, where E represents the union of all machine orders and all job orders, or by a rank matrix $A = [a_{ij}]$ (also called sequence) with specific sequence property that for each integer $a_{ij} > 1$ there exists $a_{ij} - 1$ in row i or in column j or in both (Dhamala 2007).

Our major task is to find an acyclic (feasible) combination of all machine orders (the order in which a certain job is processed on the corresponding machines) and all *job orders* (the order in which a certain machine processes the corresponding jobs), called sequence, which minimizes the maximum completion time, that is an *optimal schedule*. The set of all n × m sequences is denoted by S_{nm}. A sequence A is called reducible to another sequence B if $C_{\max}(B) \le C_{\max}(A)$ for all $P \in P_{nm}$, we write $B \preceq A$. A sequence A is called *strongly reducible* to B, denoted by $B \prec A$ if $B \preceq A$ but not $A \preceq B$.

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Two sequences A and B are called similar, denoted by $A \approx B$ if $B \preceq A$ and $A \preceq B$ hold. A sequence A is called *irreducible* if there exists no other non-similar sequence B to which A can be reduced. The irreducible sequences are the minimal sequences with respect to the partial order $\prec$ and hence are locally optimal elements. The set of all irreducible sequences contains at least one optimal solution for the problem $O \mid\mid C_{max}$ independent of the processing times. Investigations show that the ratio of all irreducible sequences to the all sequences decreases drastically as the size of the problem grows. Therefore, it is believed that the structures of these sequences would help for the development of exact or heuristic algorithms for this problem.

The problem $O_2 \mid\mid C_{max}$ is solvable in time $O(n)$ and it is NP-hard for $n \geq 3$, (Gonzalez and Sahni 1976). Braesel and Kleinau (1996), present an algorithm of the same complexity for $O_2\mid\mid C_{max}$ by means of block-matrices model. We refer to Braesel 1990, for the block-matrices model.

This dominance relation on the set of all sequences was already introduced in 1990's. The irreducible sequences for the problem $O \mid\mid C_{max}$ on an operation set with spanning tree structure and on tree-like operation sets are tested in polynomial time. This concept has been generalized by considering a dominance relation between a sequence and a set of sequences. Willenius (2000) extends the results for the other regular objective functions. Dhamala (2007) has introduced a decomposition approach in a sequence. Several necessary and sufficient conditions, which can be tested in polynomial time, and some computational results can be found in the literature (see, for instance, Braesel, Harborth, Tautenhahn and Willenius, (1999). However, up to now, no polynomial time algorithm is known for the decision whether a sequence is irreducible, in general. We refer to the references, Andresen (2009), Braesel, Harborth, Tautenhahn and Willenius (1999), Dhamala (2007), for the updated results. Andresen (2009) presents different mathematical formulations of irreducibility (reducibility) theory in the classical open shop scheduling problems (Dhamala 2010).

In this paper, we explain why H-comparability graphs constructed from classical open shop irreducible sequences are also interesting for other discrete optimization problems. Furthermore, we consider different optimization problems on H-graphs and on H-comparability graphs, discuss their relationship and the complexity status.

The paper is organized as follows. Sections 2 and 3 describe some basic properties of graph colorings and the comparability graphs, respectively. In Section 4, the properties of the comparability graphs in open shop scheduling problem are described. We construct a set of solutions for the considered problem that contains a global optimal solution for arbitrary numerical input data that is also interesting for other optimization problems on H-comparability graphs. We formulate these different optimization problems in Section 5 and present their relationships. The final section concludes the paper.

GRAPH COLORING

An undirected graph $G = (V, E)$ is called a *comparability graph*, if there exists a transitive orientation of its edges. That is, if the arcs (uv) and (vw) are contained in the orientation $D = (V, A)$, then the *transitive arc* (uw) must be also contained in $D = (V, A)$.

Comparability graphs are perfect graphs, where a graph $G = (V, E)$ is called *perfect* if, for each of its induced subgraphs G^*, the chromatic number is equal to the clique number. The *chromatic number* $\chi(G)$ of a graph $G = (V, E)$ is the smallest number of colors that can be assigned to the vertices in V such that any pair of adjacent vertices receive two distinct colors. The *clique number* $w(G)$of G is defined as the largest number of pairwise adjacent vertices in V .

By assignment of a positive integral weight $w(v)$ to each vertex v of the graph G, this property can be extended as follows: For each induced subgraph G^* of a vertex weighted comparability graph G, the weighted chromatic number $\chi_w(G^*)$ is equal to the weighted clique number $\omega_w(G^*)$. The *weighted chromatic number* $\chi_w(G)$ is the smallest number of colors for a weight coloring of the given graph, where to each vertex v, a set of colors $F(v)$ of cardinality $w(v)$ is assigned with $F(v) \cap F(w) = \phi$ for all adjacent vertices v and w. The *weighted clique number* ω_w is equal to the weight of a maximal weighted clique in the considered graph.

Vertex weighted comparability graphs are *super-perfect graphs*, i.e., the interval chromatic number $\chi_i(G)$ is equal to the weighted clique number $\omega_w(G)$. An interval coloring of G is an assignment of each vertex v to an open interval Iv of length $w(v)$ such that the intervals corresponding to adjacent vertices are disjoint. The number of colors needed for an interval coloring is the length of $\cup_v I_v$. The interval chromatic number $\chi_i(G)$ is the minimal number of colors needed for an interval coloring of G.

The calculations of all introduced chromatic numbers and clique numbers belong to *NP*-hard. However, there exist polynomial algorithms for comparability graphs. In this paper property of vertex weighted Hamming graphs and H-comparability graphs with a Hamming graph as a spanning subgraph are considered.

COMPARABILITY GRAPHS

If there exists a transitive orientation of a given graph G, then the *reserve orientation* is also transitive. We call a comparability graph *unique orientable* if only these two orientations of G are possible. Therefore, an arbitrary orientation of a randomly selected edge can be continued to a complete orientation of a comparability graph. In the literature there exist two distinct approaches for the orientations of a comparability graph which can be used to decide if a given graph is transitive orientable.

The first approach is based on the color classes or the implication classes. The transitive closure Γ^* of the following relation Γ is equivalence relation on the set of all undirected edges of the comparability graph $G = (V, E)$:

$$\forall\{ab\},\{cb\} \in E : \{ab\}\Gamma\{cb\} \Leftrightarrow \{ab\} = \{cb\} or \{ab\} \neq \{cb\} \wedge \{ac\} \notin E.$$

$$= d \wedge \{ac\} \notin E.$$

We say, the edges {*ab*}, {*cb*} form a *V*-shape, if {*ab*} Γ {*cb*} and {*ab*} $\neq$ {*cb*} is valid. The orientation of one edge forces the orientation of the second one. The generated equivalence classes are called the *color classes.*

If we set {*ab*} = {(*ab*), (*ba*)}, then the transitive closure $\Gamma^*{}_d$ of the following relation Γ_d partitions the set of edges into the equivalence classes, called the *implication classes*: $= \{cd\} or\ a\ \forall\{ab\},\{cd\}$ $\in E : \{ab\}\Gamma_d\{cd\} \Leftrightarrow \{ab\} = c \wedge \{bd\} \notin E\ or\ b$ If *A* is an implication class of a graph generated by the arc (*ab*), then the implication class A^{-1} is generated by the reverse arc (*ba*). In such a way that the set of edges is spitted into the implication classes $A_1, \ldots, A_r, A_1^{-1}, \ldots A_r^{-1}$ and any transitive orientation has to contain exactly one of each pair A_k, A_k^{-1} k = 1, ..., r. An O(n^2) time algorithm is described for the orientation of a comparability graph by means of implication classes by Simon 2000. Clearly, if {*ab*} and {*cb*} form a *V* -shape, then (*ab*) and (*cb*) belong to the same implication class. Each induced subgraph of a comparability graph is also transitive orientable. The following statements are equivalent for a graph *G* = (*V, E*) which can be used, to test, if a given graph is a comparability graph.

1. *G* is a comparability graph.

2. 2. $A_k \cap A_k^{-1} = \phi$ for all *k = 1, ..., r*.

3. *G* does not contain a closed odd walk, where no pair of vertices with distance *2* are adjacent.

4. *G* has a quasi-transitive orientation, i.e. cycles of length 3 are allowed in the orientation.

The second approach is a dual one and uses the modular decomposition of a comparability graph which generates an acyclic orientation of *G*, which is also transitive, if the graph *G* is a comparability graph. We refer to McConnell and Spinrad 2000, Dahlhaus, Gustedt and McConnel 2001, for detail description of the linear time algorithms. With this approach, the transitivity of the generated acyclic orientation has to be proved, where the time complexity increases.

The transitive closure of an acyclic oriented graph *G* is the smallest transitive oriented graph which contains *G*. The transitive reduction of a graph *G* is the smallest subgraph of *G* whose transitive closure is equal to the transitive closure of *G*. The symmetric closure of a directed graph *G* is generated from *G* by adding all arcs (*ab*) whenever (*ba*) $\in$ *E*(*G*), which makes this graph is undirected. From any given undirected graph *G* a comparability graph can be easily constructed: Calculate an acyclic orientation of *G* and determine the transitive closure of this orientation. Obviously, the symmetric closure of the obtained graph is a comparability graph.

OPEN SHOP PROBLEMS ON COMPARABILITY GRAPHS

Any sequence can be one-to-one assigned to an acyclic orientation of the Hamming graph $K_n \times K_m = (V, E)$ (called sequence graph), where two operations are connected by an edge, if they cannot be processed simultaneously, i.e., they belong to the same job or to the same machine. We describe a sequence by the rank matrix $RK = [r_k o_{ik}]$ of the corresponding sequence graph, i.e., the entry $r_k o_{ik} = l$ means that a path to operation oij with maximal number of operations has *l* operations.

If each vertex o_{ij} is weighted by its processing time p_{ij} , the time table of a semiactive schedule is given by the completion times c_{ij} of the operation o_{ij} , where c_{ij} is the weight of a maximal weighted path to operation o_{ij} . The weight of a maximal weighted path is equal to the makespan: $C_{\max} = \max\{c_{ij} / o_{ij} \in SIJ\}$. Here, we consider the open shop problem $O||C_{max}$ to minimize the maximum completion time.

We denote a simple graph as *H*-graph, if it contains a Hamming graph $K_n \times K_m$ as spanning subgraph. An *H*-graph *HG* is usually drawn into the plane as n row-cliques of size m connected to *m* column-cliques of size *n* together with diagonal edges. Therefore, $E(HG) = E(K_n \times K_m) \cup E_D$ holds, where E_D is the set of all diagonal edges. Clearly, for each Hamming graph the set E_D is empty. Furthermore, an *H*-comparability graph is an *H*-graph, which can be transitively oriented. We observe:

1. The symmetric closure of the transitive closure of a sequence graph is an H-comparability graph.
2. There exist *H*-comparability graphs with more than one sequence orientations.
3. There exist *H*-comparability graphs without sequence orientation.

An H-comparability graph *HG* has a sequence orientation, if there exist a sequence that the graph constructed by (1) yields *HG*. The investigation of *H*-comparability graphs is important in scheduling theory. All sequences obtained by different orientations of a given *H*-comparability graph have the same makespan, that is, the similar sequences which are independent from the given processing times. In the set of all irreducible sequences (potentially optimal set) there is an global optimal sequence for all processing time matrices. For more information of the irreducibility theory, we refer to Andresen 2009, Braesel and Kleinau 1996, Braesel, Harborth, Tautenhahn and Willenius 1999, Willenius 2000, Dhamala 2007, and the references therein. Note that the relation $\preceq$ generates a poset in the set of all sequences. The minimal elements of this poset are the irreducible sequences. For the sequences *A* and *B* the relation $B \preceq A$ ($B \prec A$) holds, if and only if for the corresponding comparability graphs $CG(B) \subseteq CG(A)$ ($CG(B) \subset CG(A)$) is valid, Braesel, Harborth, Tautenhahn and Willenius 1999,.

There are a number pf sufficient conditions for irreducibility of a sequence. Among them, we need in this paper a condition by means of so-called sequence implication classes, introduced by Willenius 2000. Here the relation *d* is only applied on the Hamming graph using the non-existent diagonal edges:

$$\forall \{ab\},\{cd\} \in HG : \{ab\}\gamma_d\{cd\} \Leftrightarrow \{ab\} = \{cd\} or\ a = c \wedge \{bd\} \notin E\ or\ b = d \wedge \{ac\} \notin E.$$

The transitive closure $\Gamma^*{}_d$ of this relation yields a partition of all arcs of the sequence graph in sequence implication classes. Willenius 2000 proved that a sequence is irreducible if all arcs belong to the same implication class. In particular all latin square sequences $LS[n, n, n]$ are irreducible. Note that this property is not satisfied for implication classes. Recall, a latin rectangle $LR[n, m, r]$ is an $n \times m$ matrix with entries from $B = \{1, ..., r\}$, where each element from *B* occurs at most once in each row and column, respectively. It is a latin square if $n = m = r$. In the following section, we explore how comparability graphs constructed from irreducible sequences are also interesting for other discrete optimization problems.

OPTIMIZATION PROBLEMS ON *H*-GRAPHS

In this section we consider different optimization problems on *H*-graphs and H-comparability graphs, respectively, and we discuss their relationship and their complexity status (Braesel, Bettina and Dhamala 2008). Given the Hamming graph $K_n \times K_m$ with n, m $\geq$ 2 and positive integer weight p_{ij} for each vertex v_{ij}, we formulate

Problem 1 $O \,||\, C_{max}$: Determine an acyclic orientation of this graph where the weight of a maximal weighted path (*critical path*) *Cmax* is minimal.

The calculation of C_{max} needs $O(max\{n, m)^3)$ time, because the Hamming graph contains

$m\binom{n}{2} + n\binom{m}{2}$ edges. If all weights are equal to 1, then $c_{ij} = rk(v_{ij})$.

Given the *H*-comparability graph *HG* on the Hamming graph $K_n \times K_m$ with n, m $\geq$ 2 and positive integer weight p_{ij} for each vertex v_{ij} , we formulate Problems 2 and 3.

Problem 2 *Determine the interval chromatic number* χ_i *of HG.*

Problem 3 *Determine the weight* ω_w *of a maximal weighted clique.*

It is already known that the Problems 2 and 3 for arbitrary graphs belong to *NP*-hard, even in the case of unit weights. But they are polynomial solvable for *H*-comparability graphs and it holds $\chi_i = \omega_w$.

Theorem 1 *For a fixed vertex weighted H-comparability graph HG, a maximal weighted clique and a minimal interval coloring can be calculated in polynomial time.*

Proof: The orientation of a comparability graph can be done by modular decomposition in linear time $O(|E|)$, McConnel and Spinrad 2000, which yields a complete order of all vertices. Therefore a critical path with weight c_{ij} to each vertex v_{ij} can be calculated in $O(|E|)$ time.

Because an orientation of a clique contains a Hamiltonian path, Redei 1934, the weight of a clique is equal to the weight of the contained Hamiltonian path. Therefore, the weight of a maximal weighted clique is equal to the weight of a critical path. Then a minimal interval coloring of the vertices can be constructed by $Iv_{ij} = (c_{ij} - p_{ij} , c_{ij})$ for all $v_{ij} \in V$. If all vertices have unit weights, it follows

Corollary 1 *The calculation of the clique number and the chromatic number of a fixed H- comparability graph can be calculated in linear time* $O(|E|)$.

Given the set of all *H*-comparability graphs on the Hamming graph $K_n \times K_m$ with n, m $\geq$ 2 and a positive integer weight p_{ij} for each vertex v_{ij}, we extend the Problems 2 and 3 to the Problems 4 and 5 on *H*-comparability graphs, respectively.

Problem 4 *Determine an H-comparability graph HG with minimal* χ_i *(HG).*

Problem 5 *Determine an H-comparability graph HG where* ω_w *(HG) of a maximal weighted clique is minimal.*

Theorem 2 *Consider the Problems 1, 4 and 5 with $p_{ij} = 1$ for all v_{ij}. Then the problems are polynomial solvable with optimal value max{n ,m} for all problems.*

Proof: We have to construct solutions for these problems with $C_{max} = max\{n, m\}$ and show that this value is equal to the clique number and the chromatic number. Each sequence, whose rank matrix is a latin rectangle $LR[n, m, max\{n, n\}] = [lr_{ij}]$ solves the problems which can be constructed in linear time $O(nm)$. Because we have unit weights, it holds $lr_{ij} = c_{ij}$, and therefore $C_{max} = max\{n, m\}$ is satisfied. For the comparability graph $CG(A)$ corresponding to a rank minimal sequence A, the equality $C_{max}(A) = \chi(CG(A)) = \omega(CG(A)) = max\{n, m\}$ holds, by Theorem 1.

If the weights are arbitrary, all three problems belong to *NP*-hard. Nevertheless, if one of then problems is solved, then both of the others are solved, too.

Theorem 3 *Consider the Problems 1, 4 and 5, with the same positive integer weights p_{ij}. Then there exists an optimal acyclic orientation in Problem 1 which can be one-to- one assigned to optimal H-comparability graphs in Problems 4 and 5.*

Proof: Let PO_1 and PO_2 be the partial orders on the sets of all *H*-comparability graphs on $K_n \times K_m$ with sequence orientation constructed by $\preceq$ and of all *H*-comparability graphs $HG = (V, E)$ on $K_n \times K_m$ which is given by $HG_1 \preceq HG_2$ if and only if $E(HG_1) \subseteq E(HG_2)$, respectively. Clearly, PO_1 is contained in PO_2.

Then there has to be an H-comparability graph *HG* with minimal ω_w (HG) and minimal $\chi_i(HG)$ in the set of all minimal elements in PO_2. Each orientation of such minimal *H*-comparability graph must be a sequence orientation. If there is an orientation of a minimal *H*-comparability graph, which is not a sequence orientation, at least one arc belongs to the transitive reduction in the set of all diagonal edges of *HG,* (Willenius 2000). We can cut this arc and obtain transitively oriented graph, contradicting the minimality of *HG*. In this way, Problem 1 has been embedded in Problems 4 and 5.

Because in the set of all irreducible sequences there is an optimal sequence *A* for Problem 1 independent of processing times, the corresponding comparability graph $CG(A)$ is an optimal *H*-comparability graph for the Problems 4 and 5.

An optimal *H*-comparability graph *CG* for the Problems 4 or 5 is calculated, this *H*-comparability graph is also optimal for the Problem 5 or 4, respectively, and each orientation of this H-comparability graph belongs to an optimal sequence for Problem 1. This follows directly from Theorem 1.

Recently, the theory of reducibility for the open shop problem with respect to the *H*-comparability graphs has been further investigated, (Andresen 2009, Dhamala 2010). They discuss the complexity issues of the decision problem whether a given sequence is irreducible. The results depend on the characteristics of the specific diagonal edges of the corresponding comparability graphs. It has been shown that the problem can be solved in polynomial time in most of the cases and conjectured its status for the remaining.

CONCLUDING REMARKS

The theories of reducibility and irreducibility in the classical open shop scheduling problem have been investigated since the beginning of 1990's. Since then several necessary and sufficient conditions have been established to decide whether an open shop sequence is irreducible or reducible. For instance, two machines (equivalently, two jobs) open shop problems, problems with spanning tree structure and the problems with tree-like operations sets have been solved in polynomial time.

Structural analysis of the sequence implication classes plays an important role as sequences with only one-sequence implication classes yields an irreducible sequence. Recently, a number of propositions have been made to decide its complexity status. It has been established that the critical analysis of the diagonal edges are not part of the sequence implication classes or their transitive closures play central role.

A number of conjectures have been proposed on the literature whose decisions would play decisive role on the status of the problem of reducibility. Investigations in this field are believed to develop good approximate algorithms or heuristics as the number of irreducible sequences is very small in comparison to the number of all sequences when the problem size grows.

Here in this paper, we analyzed the status of the irreducibility problem in the open shop and formulate different equivalent discrete optimization problems based on the Hamming graph, *H*-graphs and the H-comparability graphs. The investigations of this work restricted to the open shop problem with makespan objective, do have scope to the extension in the case of other shop problems like job shop scheduling problems and other general regular objective functions.

WORKS CITED

Andresen, M. 2009. *On the Complexity of Reducibility Problems through H-Comparability Graphs*. Ph.D Dissertation (in German). University of Magdeburg, Germany.

Braesel, H., M. Bettina, and T.N. Dhamala, 2008. *H-comparability Graphs: Properties and Applications in Discrete Optimization*. Preprint, University of Magdeburg, Germany.

Braesel, H., M. Harborth, T. Tautenhahn, and P. Willenius. 1999. On the Set of Solution of the Open Shop Problem. *Annals of Operations Research* 92:265-279.

Braesel H., and M. Kleinau. 1996. New Steps in the Amazing World of Sequences and Schedules. *Mathematical Methods of Operations Research* 43:195-214.

Braesel, H. 1990. *Latin Rectangle in Scheduling Theory.* Dissertation (in German). University of Magdeburg, Germany.

Dahlhaus, E., J. Gustedt, and R. McConnel. 2001. Efficient and Practical Algorithms for Sequential Modular Decomposition. *Journal of Algorithms* 4:360-387.

Dhamala, T,N. 2010. *Reducibility Problems of Open Shop Sequences Minimizing the Makespan*. 19th Workshop on Discrete Optimization, Holzhau, Germany.

--------- 2007. On the Potentially Optimal Solutions of Classical Shop Scheduling Problems. *International Journal of Operations Research (IJOR)* 4: 1-10.

Gonzalez, T., and S. Sahni. 1976. Open Shop Scheduling to Minimize Finish Times. *Journal of the Association of Computing Machinery* 23: 665-679.

McConnell R.M., and J. P. Spinrad, 2000. Ordered Vertex Partitioning. *Mathematics and Theoretical Computer Science* 4: 45-60.

R´edei, L. 1934. A Combinatorial Theorem (in German). *Szeged* 7:39-43.

Simon, K. 2000. *Efficient Algorithms for Perfect Graphs* (in German). G. Teubner, Stuttgart, Germany.

Willenius, P. 2000. *Irreducibility Theory in Scheduling Theory*. Ph.D Dissertation (in German). Shaker Verlag, Germany.