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APPLICATION OF CONTINUED FRACTION IN PELL'S EQUATION

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ABSTRACT

This paper uses a continued fraction to explain various approaches to solving Diophantine equations. It first examines the fundamental characteristics of continued fractions, such as convergents and approximations to real numbers. Using continued fractions, we can solve the Pell's equation. Certain theorems have also been discussed for how to expand quadratic irrational integers into periodic continued fractions. Finally, the relationship between convergents and best approximations and use of continuous fraction in calendar construction has been investigated. The analytical theory of continued fractions is a significant generalization of continued fractions and represents a large field for current and future research.

Keywords: continued fraction - Pell’s Equation - integer solution - convergents -approximations.

INTRODUCTION

The origin of continued fractions is typically attributed to the discovery of the Euclidean algorithm (Olds 1963). It is widely acknowledged that development of continued fractions' began with Euclid's Algorithm's introduction around 300 B.C. The greatest common divisor (gcd) between two numbers is determined using Euclid's algorithm (Szuse & Rochett 1992). It is essential to remember that the technique can be modified algebraically to obtain the continued fraction of a rational $\frac{a}{b}$, unlike the greatest common divisor of a and b. Rafael Bombelli (Olds 1963) sought to find square roots in his L'Algebra Opera by using infinite continued fractions. Rafael Bombelli found that it was possible to express the square root of 13 as a continued fraction in the 16th and 17th centuries. Whether a real number is irrational or rational, it can be expressed using continued fractions. Real number $\beta = \frac{a}{b}$, where b is not zero and both a and b are integers, are referred to as rational numbers. If β is not rational, then it is irrational (Jacobson & Williams 2009). A finite and infinite sum of the successive division of integers represents any real number as a continued fraction.

Continued fraction (Katz *et al.* 2013) is expressed as:

$$\alpha_0 + \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \cfrac{\beta_3}{\ddots + \cfrac{\beta_4}{\alpha_{n-1} + \cfrac{\beta_n}{\alpha_n}}}}} \tag{1}$$

where α_i and β_i are either rational, real, or complex numbers for every $0 \le i \le n$.

However, a methodical strategy for recognizing the expansion of any real number, not just rational ones, would be valuable. We discuss a technique for identifying numbers that have continued fraction expansion. We accomplish this using the continued fraction approach.

Continued fraction algorithm: Define $\alpha = \alpha_0$. The initial partial quotient of the continued fraction is the greatest integer less than or equal to α_0 (Sanna 2017).

$$\text{set } a_0 = \lfloor \alpha_0 \rfloor, \text{ define } \alpha_1 = \frac{1}{\alpha_0 - a_0}$$

$$\text{set } a_1 = \lfloor \alpha_1 \rfloor, \text{ define } \alpha_2 = \frac{1}{\alpha_1 - a_1}$$

set $a_2 = \lfloor \alpha_2 \rfloor$, and continue this process up to i^{th} steps. define $\alpha_i =$, Set $a_i = \lfloor \alpha_i \rfloor$, where Set $\alpha_i - a_i = 0$, find a value of $\alpha_i \in \mathbb{N}$ then we stop.

Problem: Consider the continued fraction expansion for $\frac{42}{31}$

We find $a_0 = \left\lfloor \frac{42}{31} \right\rfloor = 1$, choose $\alpha_1 = \dfrac{1}{\frac{42}{31} - 1} = \dfrac{31}{11}$

We find $a_1 = \left\lfloor \frac{31}{11} \right\rfloor = 2$, choose $\alpha_2 = \dfrac{1}{\frac{31}{11} - 2} = \dfrac{11}{9}$

We find $a_2 = \left\lfloor \frac{11}{9} \right\rfloor = 1$, choose $\alpha_3 = \dfrac{1}{\frac{11}{9} - 1} = \dfrac{9}{2}$

We find $a_3 = \left\lfloor \frac{9}{4} \right\rfloor = 4$, choose $\alpha_4 = \dfrac{1}{\frac{9}{2} - 4} = \dfrac{2}{1}$

We find $a_4 = \left\lfloor \frac{2}{1} \right\rfloor = 2$, it follows that $\alpha_4 - a_4 = 2 - 2 = 0$. We are done.

We have, $\frac{42}{31} = [1; 2, 1, 4, 2]$. This shows that the algorithm stops after finitely many steps. So, this is a reasonable approach to express all rational numbers. The formalities are as follows;

If $\alpha_r \in \mathbb{R}$ for $r = 0,1,2, ..., r$, where $r \in \mathbb{N}$, and $a_r \in \mathbb{R}^+$ for $r > 0$, then an expression of the form:

$$\beta = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{r-1} + \cfrac{1}{\alpha_r}}}}} = [\alpha_0; \alpha_1, \alpha_2, \cdots, \alpha_r]$$

is a continued fraction with a length r. If $\alpha_i \in \mathbb{Z}$ for every $i = 0,1,2, ...,$ r a finite continued fraction is called simple. The partial quotients are represented by the values α_i. The partial quotients, namely $\alpha_0 = \lfloor \alpha \rfloor$, are used to represent the floor of a real number. Every rational integer has a finite simple continued fraction expansion, as demonstrated by the Euclidean algorithm (Rosen 1984). First, Euler (Mollin 1998) established that a real number is rational if and only if its continued fraction expansion is finite. As a result, every rational number can be expressed as a finite continued fraction, and vice versa.

Theorem 1: A rational number can be represented as a finite continued fraction, and vice versa. In addition, this continued fraction is distinct, aside from the identity $[\alpha_0; \alpha_1, \alpha_2, \ldots, \alpha_n] = [\alpha_0; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 1]$ (Olds 1963).

If $\alpha_r \in \mathbb{R}$ where $r \in \mathbb{N}$ and $\alpha_r \in \mathbb{R}^+$ for $r > 0$, then an expression of the form

$$\beta = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{r-1} + \cfrac{1}{\alpha_r + \ddots}}}}} = [\alpha_0; \alpha_1, \alpha_2, \cdots, \alpha_r, \alpha_{r+1}, \ldots]$$

is an infinite continued fraction. The infinite continued fraction is referred to as simple if $\alpha_r \in \mathbb{Z}$, $r \geq 0$. As a result, every infinite continued fraction is irrational, and there is only one way to express every irrational number as an infinite continued fraction. An infinite continued fraction representation for an irrational integer is useful because its starting segments provide reasonable approximations of the number.

Problem: If irrational numbers are expressed as continued fractions, they are non-terminating. So, using continued fractions, the irrational number π can be expressed in the form:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}}}}}} = [3; 7, 15, 1, 292, 1, 1, 1, 2, \cdots]$$

Using the continued fraction, we can estimate how effectively rational numbers can approximate irrational numbers. The convergent of the continued fraction refers to these rational numbers.

Convergents of the continued fraction: When $C = [\alpha_0; \alpha_1, \alpha_2, \ldots, \alpha_k]$ is the continued fraction, then the convergents of the continued fraction are

$$C_0 = \alpha_0$$

$$C_1 = \alpha_0 + \frac{1}{\alpha_1}$$

$$C_2 = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2}}$$

$$\ddots$$

$$C_k = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{k-1} + \cfrac{1}{\alpha_k}}}}}$$

is referred to as the k^{th} convergents to C (Watkins 2013). Suppose $C = [\alpha_0; \alpha_1, \alpha_2, \ldots, \alpha_k]$ and we define

$$p_n = a_n p_{n-1} + p_{n-2}, p_{-2} = 0, p_{-1} = 1, p_0 = a_0 \text{ and}$$
$$q_n = a_n a_{n-1} + q_{n-2}, q_{-2} = 1, q_{-1} = 0, q_0 = 1$$

Then the following properties are held

- convergent $C_n = \frac{p_n}{q_n}$
- $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ and $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$, for all $n \geq 0$
- $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$ and $C - C_{n-2} = \frac{(-1)^{n-1} a_n}{q_{n-2} q_n}$
- $C_1 > C_3 > C_5 > \cdots > C_6 > C_4 > C_2$ and $|C - C_{n-1}| \leq \frac{1}{q_n q_{n-1}} < \frac{1}{q_n^2}$

The convergents of the continued fractions (Jacobson & Williams 2009) are defined as $\alpha_0, \alpha_0 + \frac{1}{\alpha_1}, \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2}}, \cdots$ resulting from truncating the fraction at a term that is earlier than α_n. Note that we truncate at α to obtain the following to define the general continued fraction

$$C_k = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_{k-1} + \cfrac{1}{\alpha_k}}}}} = \frac{A_k}{B_k} = [\alpha_0, \cdots, \alpha_k]$$

Theorem 2: The k^{th} convergents of the continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ has $C_k = \frac{P_k}{Q_k}$, for $0 \le n \le k$, where $p_k = a_k p_{k-1} + p_{k-2}$, $q_k = a_k q_{k-1} + q_{k-2}$. If the chain in equation (1) does not stop, then we define the relation as an infinite continued fraction. It is convergent if the limit of C_k exists and is equal to β, that is $\lim_{k\to\infty} C_k = \beta$. A continued fraction is defined as periodic if it is non-terminating and for some given integers k and r the terms $\alpha_k; \alpha_{k+1}, \ldots, \alpha_{k+r-1}$, repeat infinitely in the expansion and the continued fractions are said to have a period r. We can write $[\alpha_0; \alpha_1, , \ldots, \overline{\alpha_k, \alpha_{k+1}, \cdots, \alpha_{k+r-1}}]$. In, the convergents are the best rational approximations of ρ if $\frac{p_k}{q_k}$ is a convergent to $\rho = [\alpha_0; \alpha_1, \alpha_2, \ldots,]$, then $| \frac{p}{q} - \rho | < \frac{1}{2q^2}$ if and only if then $\frac{p}{q}$ is a convergent to ρ (Watkins 2013).

METHODS AND MATERIALS

When applicable, the main objective is to solve Pell's equation using continued fractions. The main conclusion of the necessity for the application of continued fractions into Pell's Equation is established after examining and discussing the previously available materials. The statement is supported by theorems and examples from the study of number theory in a descriptive way. Therefore, it will be our responsibility to identify and thoroughly understand the continued fraction expansion of $\sqrt{d}$.

In considering the following definition, it will come out that these expansions are periodic (Panda 2011).

Periodic: If a continued fraction is non-terminating and the terms $a_k, \cdots, a_{k+r-1}$ repeat infinitely in the expansion for some given integers k and r, are said to be periodic and to be a period r. It is denoted by $a_0, a_1, \cdots, a_k \cdots, a_{k+r-1}$.

Purely periodic: A continued fraction is referred to as being purely periodic with period m if the initial block of partial quotient $a_0; a_1, a_2, \cdots, a_{m-1}$ repeats infinitely, and no block of length less than m is repeated. When a continued fraction has an initial block of length n and a repeating block of length m, the continued fraction is said to be periodic with period m. It is denoted by $[\overline{a_0; a_1, a_2, \cdots, a_{m-1}}]$.

We shall first discuss periodic continued fractions in general before focusing on the continued fraction expansion of $\sqrt{d}$ specifically. The set of quadratic irrational integers comes out to be exactly as:

Quadratic irrational: Assume that β is a real number. If a number β is irrational and $\beta = u\sqrt{d}+v$, where u, v are rational integers and $d > 0$, not perfect square integer, then that number is said to be a quadratic irrational number.

RESULTS AND DISCUSSION

It is possible to approximate irrational numbers by rational numbers by using continued fractions. Determining the rational number with the lowest positive denominator among those that differ from irrational numbers by no more than a specific value is a part of the approximation problem. Using a smaller numerator and denominator, this method is also used to approximate rational numbers; whose numerator and denominator are extremely large.

Theorem 3: Suppose that $\alpha = [\overline{a_0; a_1, a_2, \cdots, a_{n-1}, a_n}]$ is purely periodic continued fraction consequently, $\frac{-1}{\alpha'} = [\overline{a_n; a_{n-1}, \cdots, a_1, a_0}]$, where α' is conjugate of α (Olds 1963).

Theorem 4: Suppose two integers p and q such that $p > q > 0$. Then $[a_0; a_1, a_2, \cdots, a_{n-1}, a_n]$ is the continued fraction of $\frac{p}{q}$ if and only if $\frac{q}{p}$ has $[0, a_0; a_1, a_2, \cdots, a_{n-1}, a_n]$ as its continued fraction (Szuse & Rochette 1992).

We must be informed of the continued fraction expansion of $\sqrt{d}$ to solve Pell's equation using continued fractions. Such an expansion is provided by the following theorem 5:

Theorem 5: Suppose $d > 0$ is not a perfect square. Then expansion of $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_{n-1}, a_n, 2a_0}]$, where $a_{n+1-j} = a_j$ for $j = 1,2, \cdots, n$ (Olds 1963).

In other words, $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_2, a_1, 2a_0}]$

Proof

If $\sqrt{d} > 1 \Rightarrow -\sqrt{d} < -1$, then $\sqrt{d}$ is not reduced quadratic irrational.

Suppose

$$\sqrt{d} = [a_0; a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{3}$$

Since $\sqrt{d} > 1$, is not a perfect square, then $a_0 + \sqrt{d} > 1$. We have, $0 < \sqrt{d} - a_0 < 1 \Rightarrow -1 < a_0 - \sqrt{d} < 0$ is a conjugate of $a_0 + \sqrt{d}$ that lies between – 1 and 0. So, $a_0 + \sqrt{d}$ is a reduced quadratic irrational and it has a purely periodic continued fraction

We add a_0 in (3)

$$a_0 + \sqrt{d} = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Since the expansion of a reduced quadratic irrational is purely periodic. Then

$$a_0 + \sqrt{d} = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}} = \left[2a_0; \overline{a_1, a_2, \ldots, a_n}\right]$$

$$\Rightarrow \sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}} = \left[a_0; \overline{a_1, a_2, \ldots, a_n, 2a_0}\right]$$

By theorem 2, gives

$$\frac{-1}{a_0 - \sqrt{d}} = \frac{1}{\sqrt{d} - a_0} = \left[\overline{a_n, \ldots, a_1, 2a_0}\right]$$

Also, by subtracting a_0 from equation (3), we have,

$$\sqrt{d} - a_0 = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [\overline{0; a_1, a_2, \cdots, a_n, 2a_0}] \tag{4}$$

By theorem 3, it gives

$$\frac{1}{\sqrt{d} - a_0} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = \left[\overline{\mathrm{a}_1, a_2, \cdots, a_n, 2a_0}\right] \tag{5}$$

Comparing equations (4) and (5), we have,

$a_n = a_1$, $a_{n-1} = a_2$, $\cdots$, $a_2 = a_{n-1}$, $a_1 = a_n$

Hence $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_2, a_1, 2a_0}]$.

The subject of continued fractions seems to be almost unknown to the Western students of modern Mathematics. However, it plays a significant, if not essential, role in both analytic and arithmetic theory. Numerous relevant areas of study can benefit from the use of continued fractions in Pell's Equation, such as the theory of equations, power series, quadratic forms in infinitely many variables, definite integrals, analytic functions, and the summation of divergent series.

Wall's analytic framework of continued fractions was introduced in 1948 and served as a conceptual model for continued fractions and their applications (Wall 1948). This excellent work is highly recognized as the definitive text on the analytic theory of continuous fractions.

Continued fractions can be used in a variety of situations, including Pell's Equations and the determination of fundamental units in quadratic fields, the reduction of quadratic forms, and the determination of class numbers for quadratic fields. Continued fractions can also be used in conjunction with the golden ratio and Fibonacci numbers (Rosen 1984). There is a connection between Chebyshev polynomials, Pell's Equation, and continued fractions because the latter two are thought to occur in real quadratic function fields, as opposed to the more traditional case of real quadratic number fields (Barbeau 2003). In Mathematics, the simple continuous fractions have been examined using Pell's equations, the Diophantine equation, and the congruence ax ≡ b (mod m) can also be solved using simple continued fractions. Wall (1948) talks about the study of continued fractions for real and complex values. Wall established that any number, whether rational or irrational, can be expressed as a finite or infinite continuing fraction.

Application of continued fraction in Pell’s Equation: It is thought that the square root of square-free positive integers was first approximated using continued fractions. We present important information about continued fractions, which we will use to solve specific Pell's Equations.

Theorem 6: Let $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be an irreducible polynomial with integral coefficients and degree of $n \geq 3$. Let us consider the homogeneous polynomial

$$
\begin{aligned}
F(x, y) &= y^n f\left(\frac{x}{y}\right) \\
&= a_n x^n + a_{n-1} x^{n-1} y + \ldots + a_1 x y^{n-1} + a_0 y^n
\end{aligned}
$$

Then the equation $F(x, y) = N$ has either no solution or only a finite number of solutions in integers (Thue 1909).

In the theorem 6 is in contrast when the degree of F is $n = 2$ (Dickson 1957).

For instance, if $F(x, y) = x^2 - dy^2$, where $d > 1$, is not a perfect square, then for non-zero integer N, the quadratic Diophantine equation of the form

$$x^2 - dy^2 = N \tag{6}$$

has either no integral solutions or infinitely many solutions, which is known as the generalized Pell's Equation (Dickson 1957) after John Pell, a Mathematician who studied in the *17^th^* century to find the integer solutions to equation (6).

Theorem 7: If $d > 1$, is not a perfect square integer, then $h_n^2 - dk_n^2 = (-1)^{n-1} q_{n+1}$ for every integer $n \geq -1$ (Kumundury & Romero 1998).

Theorem 7 gives us a solution to (6) for a given value of N. So, the following theorem 8 establishes the connection between the convergence of $\sqrt{d}$ and the solutions of equation (6) for $0 < N < \sqrt{d}$.

Theorem 8: Let $0 < N < \sqrt{d}$ and (P, Q) be a solution of the equation $x^2 - dy^2 = N$. Then $\frac{u}{v}$ is convergent in the expansion of $\sqrt{d}$ (Niven *et al.* 1991).

Proof

Since (u, v) is a solution of the equation (6), then

$N = u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d})$

Since $0 < N < \sqrt{d} \Rightarrow \sqrt{d} > N > 0$

Then, we have

$$0 < \frac{u}{v} - \sqrt{d} < \frac{N}{v\left(u + v\sqrt{d}\right)} < \frac{\sqrt{d}}{v\left(u + v\sqrt{d}\right)}$$

Since $v\sqrt{d} < u \Rightarrow 2v\sqrt{d} < u + v\sqrt{d}$

Then, we have

$$0 < \frac{u}{v} - \sqrt{d} < \frac{\sqrt{d}}{v\left(u + v\sqrt{d}\right)} < \frac{\sqrt{d}}{2v^2\sqrt{d}} = \frac{1}{2v^2}$$

$$\Rightarrow 0 < \frac{u}{v} - \sqrt{d} < \frac{1}{2v^2}$$

$$\Rightarrow \left|\frac{u}{v} - \sqrt{d}\right| < \frac{1}{2v^2}$$

It follows that $\frac{u}{v}$ is a convergent of $\sqrt{d}$.

Theorem 9: If $|N| < \sqrt{d}$, then the solutions of equation $x^2 - dy^2 = N$ are $x = u_n, y = v_n$, where $\frac{u_n}{v_n}$ is a convergent of $\sqrt{d}$.

Problem: Observe Pell's equation $x^2 - 7y^2 = 2$. Since $2 < \sqrt{7}$, we know that the solution (p_n, q_n) is a convergent $\frac{p_n}{q_n}$ of the continued fraction expansion of $\sqrt{7}$. So, continued fraction expansion of $\sqrt{7} = [2; \overline{1,1,1,4}]$.

When $N = \pm 1$, the Diophantine equation (6) becomes

$$x^2 - dy^2 = \pm 1 \tag{7}$$

It is known as Pell's Equation, where $d > 0$, is not a perfect square. Pell's Equation uses a straightforward algebraic approach with a finite number of solutions when d>1 and a perfect square. Pell's Equation can be used to solve a variety of problems because it always has the trivial solution $(x, y) = (\pm 1, 0)$ and has an infinite number of solutions. The Indian mathematicians Brahmagumpta and Bhaskara developed techniques for resolving Pell's equations (Barbeau 2003). Pell's Equation can be resolved using the Chakravala method, which Brahmgupta first developed. These equations were used in the time of Pythagoras to approximate the square root of 2 (Pang 2011). So, Pell's Equation is also known as the classical Pell's Equation (Barbeau 2003, Niven *et al*. 1991) and Brahmagupta and Bhaskara were the first to study Pell's equation (Arya 1991).

The theory was developed by Lagrange, not Pell. Lagrange was the first to establish that there are infinitely many solutions to Pell's Equation, if d is a positive, not a perfect square (Legendre 1798). The Indian mathematician Baudhayana discovered in the fourth century that the equation $x^2 - 2y^2 = 1$ has a solution $(x, y) = (577,408)$, and he used the ratio $\frac{577}{408}$ to approximate $\sqrt{2}$.

But $\frac{577}{408} \approx 1.4142156$, while $\sqrt{2} = 1.4142135$ Archimedes estimated $\sqrt{3} \approx 1.7320508$ by $\frac{265}{153} \approx 1.7320261$ and $\frac{1351}{780} \approx 1.7320512$, then $\frac{x}{y}$ satisfy the equations $x^2 - 3y^2 = -2$ and $x^2 - 3y^2 = 1$. The smallest solution $(x, y) = (1151,120)$ to Pell's Equation $x^2 - 91y^2 = 1$, was investigated by

Brahmagupta in the seventh century. Similarly the least-positive solution $(x, y) = (1776319049, 2261590)$ to Pell's equation $x^2 - 61y^2 = 1$ was given by the Hindu mathematician Bhaskara in the twelve century.

Therefore, there are always infinitely many possible solutions to Pell's Equation (7). They can be found by continued fraction expansion of $\sqrt{d}$. The fundamental solution of equation (7) is usually the least positive solution. The following theorem 8 shows that if (x_1, y_1) is the fundamental solution to equation (7) then there are infinitely many solutions, and they are all generated from (x_1, y_1).

Another application of Pell's Equation is the approximation of square roots. Suppose that (x, y)satisfies Pell's equation. We cannot write , $\sqrt{d} = \frac{x}{y}$ where $x, y \in \mathbb{Z}$, $\sqrt{d}$ is irrational.

But, if $x^2 - dy^2 = 1 \Rightarrow \frac{x^2}{y^2} = d + \frac{1}{y^2} \approx d$. Therefore, Pell's solutions result in accurate rational approximations of $\sqrt{d}$. As a result, for large y, $\frac{x}{y}$ is a good approximation to $\sqrt{d}$. Therefore There are non-trivial solutions and infinitely many solutions to Pell's equation $x^2 - dy^2 = 1$. The fundamental solution, which is generated by Theorem 10, is at least one convergent of $\sqrt{d}$ and yields all solutions.

Theorem 10: Suppose $d > 0$ is not a perfect square. Then the continued fraction expansion of $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_{r-1}, 2a_0}]$, where r is the length of period, then the fundamental solution (x_1, y_1) to Pell's equation (7) is given by the continued fraction expansion $\frac{x_1}{y_1} = [a_0; a_1, a_2, \cdots, a_{r-1}]$. Define $\frac{x_n}{y_n} = [a_0; a_1, a_2, \cdots, a_{nr-1}]$, then $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, for integer $n \geq 1$, $n \in \mathbb{Z}$ (Hoffstein *et al.* 2008).

Theorem 11: If (x_1, y_1) is the fundamental solution to Pell's equation (7), then n^{th} positive solution is (x_n, y_n), where x_n and y_n are given by $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, for integer $n > 1$, $n \in \mathbb{Z}$ (Waldschmidt 2016), which leads us to the following explicit form;

$$x_n = \frac{1}{2}\left\{\left(x_1 + y_1\sqrt{d}\right)^n + \left(x_1 - y_1\sqrt{d}\right)^n\right\},$$

$$y_n = \frac{1}{2\sqrt{d}}\left\{\left(x_1 + y_1\sqrt{d}\right)^n - \left(x_1 - y_1\sqrt{d}\right)^n\right\}$$

In addition, the solutions (x_n, y_n) satisfy the recurrence relations

$$x_{1+n} = 2x_1 x_n - x_{n-1}, \qquad y_{1+n} = 2x_1 y_n - y_{n-1}$$

$$x_{1+n} = x_1 x_n + y_1 y_n d, \quad y_{1+n} = x_1 y_n + y_1 x_n$$

Theorem 12: Let $d > 0$, not a perfect square, and $\frac{p_n}{q_n}$ be the n^{th} convergent of $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_{r-1}, a_r}]$, where r is length of period.

All positive solutions of $x^2 - dy^2 = 1$are given by

$$(x, y) = \begin{cases} (p_{kr-1}, q_{kr-1}), k \in \mathbb{N}, if\ r\ is\ even \\ (p_{2kr-1}, q_{2kr-1}), k \in \mathbb{N}, if\ r\ is\ odd \end{cases}$$

All positive solutions of $x^2 - dy^2 = -1$ are given by

$$(x, y) = \begin{cases} (p_{kr-1}, q_{kr-1}), k \in \mathbb{N}, if\ r\ is\ odd \\ \quad no\ solution\ if\ r\ is\ even \end{cases}$$

Moreover, (p_{r-1}, q_{r-1}) is a fundamental solution of

$$\begin{cases} x^2 - dy^2 = 1, if\ r\ is\ even \\ x^2 - dy^2 = -1, if\ r\ is\ odd \end{cases}$$

and (p_{2r-1}, q_{2r-1}) is the fundamental solution of $x^2 - dy^2 = 1$ if r is odd (Niven *et al*. 1991).

We found a fundamental solution and used the fundamental solution to find other positive integral solutions to Pell's Equation.

Problem: Solve Pell's equation $x^2 - 41y^2 = 1$ using the method of continued fractions.

We begin by computing the continued fraction expansion of $\sqrt{41} = [6; \overline{2,2,12}]$. It has a length of period $r = 3$, which is odd. Therefore, negative Pell's Equation $x^2 - 41y^2 = -1$ has a solution. So, 3^{rd} convergent is

$C_2 = 6 + \cfrac{1}{2 + \cfrac{1}{2}} = 6 + \frac{2}{5} = \frac{32}{5}$. Thus, $(x, y) = (32,5)$ is a solution to the negative Pell's Equation.

Problem: Indian Mathematician Brahmagupta asked to finda fundamental solutionand other solutions of Brahmagupta equation $x^2 - 92y^2 = \pm 1$ by using a continued fraction algorithm.

First of all, the continued fraction expansion of $\sqrt{92} = [9; \overline{1,1,2,4,2,1,1,18}]$. So, the length of the period of the continued fraction is $r = 8$, which is even. Hence, the required fundamental solution is $(x_0, y_0) = (p_{r-1}, q_{r-1}) = (p_7, q_7) = (1151,120)$.

We also consider an application of continued fractions to make the calendar problem (Euler 1748),

Problem: One year lasts exactly 365 *days, five hours, eight minutes, and fifty-five seconds*, according to precise astronomical observations (Sanna 2017). But the calendar didn't show any obvious errors for a long period. A mistake of 5 hours is produced annually by the assumption that a year has 365 days. Seasonal changes become apparent after 100 years due to the mistake, which builds up rather quickly. The contradiction with the seasons will be seen much earlier if we suppose that a year has 366 days.

We first convert one year into days to solve this problem:

$$\begin{aligned} 1\text{ Year} &= 365 + \frac{5}{24} + \frac{48}{60} \times \frac{1}{24} + \frac{55}{60} \times \frac{1}{60} \times \frac{1}{24}\text{day} \\ &= 365 + \frac{20935}{86400}\text{days} \end{aligned}$$

This period is approximate, but the mistake is so minor that it won't be apparent for at least 1,000 years. Now, we assert that $\frac{20935}{86400}$ is a reasonable approximation.

This rational integer is transformed into a regular continued fraction. So 20935 0 and 86400 are both divisible by 5, making $\frac{20935}{86400} = \frac{4187}{17280}$. The final fraction is in the lowest terms, which is simple to explain the fact that $20935 = 5 \times 53 \times 79$ whereas $86400 = 2^7 \times 3^3 \times 5^2$, suggests that 5 is the greatest common divisor. By continued fraction expansion of

$$\frac{4187}{17280}=0+\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\ddots}}}}}}}}}=[0;4,7,1,6,1,2,2,4,2,\cdots]$$

Computer systems for computing rational number approximations to real numbers use continued fractions. The fields of cryptography and hyperbolic geometry have both adopted continued fractions.

CONCLUSIONS

Throughout this paper, we have investigated simple continued fractions and explored an application of solving Pell's Equation. We have accomplished our main purpose of solving Pell's Equation. There are some applications of the continued fraction in the solutions of Pell's Equation. Continued fractions are not the optimal tool to approximate square roots. In modern Mathematics, continued fractions are essential. They are an essential tool for understanding developments in the areas of Diophantine approximations and number theory. The analytical theory of continued fractions is a significant generalization of continued fractions and represents a large field for current and future research. When designed for electronic devices, continued fractions are used in the computer industry to approximate a variety of complex functions and provide quick numerical results that are useful for scientists and researchers working in applied Mathematics.

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