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From Rule to Reason: Grade 8 Students' Understanding of the Distributive Property under the Field Axioms of Real Numbers

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Abstract

This qualitative phenomenological study explored how Grade 8 students in Nepal conceptualize and apply the field axioms of real numbers, with particular attention to the distributive law. Data were collected through classroom observation, task-based interviews, and students' written work from fifteen instructional sessions. Thematic analysis revealed five major patterns of understanding. Most students demonstrated procedural fluency without conceptual depth, relying on memorized steps such as "multiply both" rather than logical reasoning. Many confused distributions with expansion, viewing factorization as a division process instead of a reversible property of equivalence. The connection between the axiomatic structure of real numbers and their operations was poor, as students performed operations correctly but could not relate them to closure, commutativity, or identity. The teacher-centered classroom culture reinforced procedural learning and limited opportunities for reflective dialogue. Nevertheless, a few students showed emerging conceptual shifts when supported by visual and reflective activities, linking distributivity to area models and recognizing its reversibility. The study concludes that explicit engagement with axiomatic reasoning and dialogic instruction can help students move from instrumental to relational understanding. It recommends integrating visual representations, reasoning prompts, and reflective discussion to strengthen structural comprehension of the field axioms in lower-secondary mathematics.

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Introduction

Mathematics teaching activities at the school level are deeply rooted in the axiomatic structure of real numbers. The system of logic and relationships guided by axioms govern how numbers and operations interact. Among these, the field axioms of the real number are fundamentals of arithmetic and algebra. The set of real numbers, denoted by R together with two binary operations addition (+) and multiplication (×) satisfies: closure axioms $[\forall\, a, b \in R \text{ then } a + b, a \times b \in R]$, commutativity $[\forall\, a, b \in R \text{ then } a + b = b + a,\ a \times b =\in b \times a]$, associativity $[\forall\, a, b \in R, a + (b + c) = (a + b) + c \text{ and } a \times (b \times c) = (a \times b) \times c\,]$, existance of identify [there exist $0,\ 1 \in R$ such that $a + 0 = a = 0 + a\ \&\ 1 \times a = a = a \times 1\ \forall\, a \in R$], existence of inverse [if $a \in R$ then $\exists\,(-a) \in R\ such\ that\ a + (-a) = 0\ and\ if\ a \in R\ \exists \frac{1}{a} = a^i, called\ inverse\ of\ a\ such\ that\quad a \times a^i = 1, \forall\, a \in R$] and distributivity [multiplicative distributivity over addition: $\forall\, a, b, c \in R, a \times (b + c) = a \times b + a \times c \text{ and } (a + b) \times c = a \times c + b \times c$] (Artigue, 2013; Hart, 2020). When complemented by the order and completeness axioms, the system of real numbers forms a complete ordered field, a structure that supports continuous reasoning essential for real analysis (Lafferriere et al., 2020; Tall, 2013). This continuity principle allows learners to extend discrete numerical intuition toward the understanding of limits, continuity, and convergence. Hence, the field axioms serve as the foundation for performing mathematical operations and developing conceptual understanding in school mathematics. Their pedagogical implication lies in nurturing students' appreciation of the logical coherence of number systems, ensuring that mathematical instruction progresses from procedural proficiency to structural understanding.

The axioms provide a coherent framework for logical reasoning, making them not only algebraic rules but also epistemic foundations for all higher mathematical thought. For example, the distributive law a(b+c)=ab+ac bridges additive and multiplicative structures, enabling students to move from concrete arithmetic to abstract algebraic generalization (Kieran, 1992; Sfard, 2008). The field axioms of the real number system work as significant tools that represent learners' ability to simplify, express, solve numerical and algebraic problems, and construct a proof

However, in most school contexts, field axioms are not explicitly discussed. They are embedded indirectly in procedural instruction, where emphasis is on rule application rather than on the logical principles that justify such rules (Booth, 1984; Artigue, 2013). As a result, students perform correct operations in familiar tasks yet struggle when problems deviate from routine forms (Akhtar et al., 2020). Explicit attention to field axioms could transform procedural fluency into conceptual understanding, allowing learners to recognize that operations are governed by general laws rather than arbitrary conventions (Tall, 2013).

Abstraction and the development of mathematical reasoning are other important factors in mathematical activities. The movement from arithmetic to algebra requires students to shift from operational to structural thinking. Sfard (2008) described this transition as one from process to object conception, while Gray and Tall (1994) characterized it as perceptual thinking, the ability to coordinate procedure and concept simultaneously. Mastery of field axioms embodies this shift, as students must recognize that each algebraic manipulation expresses a relationship grounded in consistent structural logic.

Developing such abstract reasoning requires guided progression. Tall (2013) proposed that mathematical thinking evolves through embodied, symbolic, and formal stages. First, students engage with activities with numerical ideas concretely, then they utilize symbols to express patterns, and finally, they reason within an axiomatic system. Likewise, Margulieux et al. (2021) stated that learners internalize abstract concepts through multiple-conception experiences. To gain multiple experiences, students engage in comparing and contrasting examples, which provoke reflection and reconciliation. These theories converge on the view that abstraction must be built through inquiry, dialogue, and justification, not merely through exposition.

Misconceptions are one challenge for valid reasoning and solving mathematical problems. Empirical research reveals that students often misunderstand distributive, associative, and inverse properties. Linchevski (1995) introduced the term *symbolic overload* to describe how learners treat algebraic notation as a sequence of procedural cues rather than as a representational system of general relationships. Common errors include misapplying distributivity in subtraction, for instance, writing a–(b+c)=a–b+ca – (b + c) = a – b + ca–(b+c)=a–b+c, or assuming additive and multiplicative inverses behave identically (Sfard, 2008). Al-Rabab'ah et al. (2020) reported that secondary students most frequently err in sign manipulation and distribution, showing a weak understanding of the logical basis of operations. Similarly, Ay (2017) found that such misconceptions persist despite repeated instruction, underscoring that rote learning alone cannot yield structural comprehension.

These difficulties create operational schemas that rule memorization without understanding the concept (Booth, 1984). When learners follow steps based on what they memorize (mechanically), they poorly reflect on why a rule applies; consequently, this process makes mathematical knowledge hard, and that knowledge and memorization process does not work in novel contexts. This challenge becomes particularly acute as students encounter advanced topics like real analysis, where axiomatic reasoning underpins all conceptual progression.

Different abstraction conditions in learning mathematics may have different cognitive challenges. Learning field axioms effectively depends on several cognitive and pedagogical conditions. First, students need prior fluency in connecting arithmetic and algebra so they can perceive addition and multiplication as interrelated rather than independent. Second, instruction should provide scaffolded exposure through visual and concrete models, such as area representations or algebra tiles, to make axiomatic properties tangible. Third, metacognitive involvement is necessary: students must say what property justifies each algebraic manipulation. Fourth, participants should be provided with mind-shift activities that enable them to test and rectify misconceptions that promote reflection (Margulieux et al., 2021). Without these scaffolds, students can have what Sfard (2008) referred to as structural blindness-a failure to see mathematics as a related whole influenced by general rules. This inability results in a piecemeal comprehension and enduring procedural reliance, particularly concerning abstract thought-based topics.

In Nepal, mathematics instruction at the lower-secondary level remains dominated by procedural teaching and exam-oriented assessment. Teachers emphasize correct answers and mechanical steps over conceptual explanation (Bhattarai & Basnet, 2022). Although the Mathematics Curriculum for Lower Secondary Level (Curriculum Development Centre, 2021) lists distributive, associative, and commutative properties, they are presented as formulas rather than as derivable axioms. Consequently, students may memorize a(b+c)=ab+ac yet be unable to explain why it is valid or reversible. Teachers rarely highlight the logical relationships among the axioms or connect them to the real-number system's structure.

Empirical work from South Asia confirms similar trends. Akhtar et al. (2020) reported that students in Nepal and northern India often experience confusion when transferring arithmetic understanding to algebraic expressions. The lack of manipulatives, dialogic discourse, and inquiry-based learning reinforces procedural competence without conceptual depth. Observations from Grade 8 classrooms reveal recurrent errors such as 2(x+3) =2x+3 or a(b–c) =ab–c, indicating incomplete internalization of distributivity. These errors illustrate how students' reasoning remains grounded in pattern imitation rather than in the relational logic of the field axioms.

Addressing students' conceptual difficulties with field axioms is essential for improving the quality of mathematical understanding and instruction in Nepal, and making these axioms explicit can transform learners' procedural fluency into relational reasoning and can nurture reflective habits of thought fundamental to higher mathematics. As Tall (2013) noted, the progression from computational to axiomatic reasoning marks a developmental threshold in mathematical maturity. Strengthening conceptual understanding of field axioms thus aligns with global educational goals that emphasize reasoning, justification, and transfer over rote performance (Sfard, 2008; Margulieux et al., 2021). Guided by these concerns, the present study investigates how Grade 8 students in Nepal understand, interpret, and apply the field axioms, particularly the distributive property, in mathematical problem-solving contexts. It seeks to identify the nature of students' misconceptions, the cognitive obstacles underlying them, and the instructional conditions that foster or hinder abstraction. The following research questions guide the study:

1. How do Grade 8 students conceptualize and apply the field axioms, especially the distributive law, in mathematical problem solving?
2. What specific misconceptions and reasoning patterns emerge in students' understanding of field properties?
3. What classroom practices and learning conditions influence students' abstraction and structural comprehension of the field axioms in the Nepali context?

Theoretical Framework

Learning theories and theories related to knowledge transformation were reviewed to understand the knowledge construction process and knowledge transformation. The theoretical framework was developed based on constructivist learning theory, Skemp's model of understanding, Habermas's theory of knowledge interests, Tall's cognitive worlds of mathematics, and Sfard's communicative view of abstraction.

Constructive learning theory believes that learners actively construct knowledge through interaction with their environment and reflection on experiences (Piaget, 1970; Galsersfeld, 1995). The cognitive reconstruction combining prior arithmetic understanding into more abstract algebraic and logical form involves understanding mathematical properties like the distributive property of R. When learners recognize operations like a (b+c) = ab + bc, which express invariant relationships rather than isolated rules, they demonstrate structural rather than procedural knowledge. Hence, constructivism thus situates learning as a process of meaning-making where knowledge evolves through engagement and self-regulation.

Skemp's (1976) distinction between instrumental and relational understanding further clarifies this developmental process. Instrumental understanding represents rule-following competence without comprehension of the underlying rationale, while relational understanding reflects the ability to connect mathematical ideas meaningfully. This theoretical lens helps reveal the nature of students' reasoning rather than merely their performance.

To extend this cognitive analysis, Habermas's (1971) theory of knowledge-constitutive interests provides a reflective dimension that aligns with different levels of mathematical understanding. The technical interest corresponds to procedural accuracy, the practical interest relates to interpretive comprehension, and the emancipatory interest signifies reflective abstraction and autonomy. When students move from applying distributivity mechanically (technical) to explaining its meaning (practical) and finally to reflecting on its logical necessity (emancipatory), they transition from procedural fluency to conceptual independence.

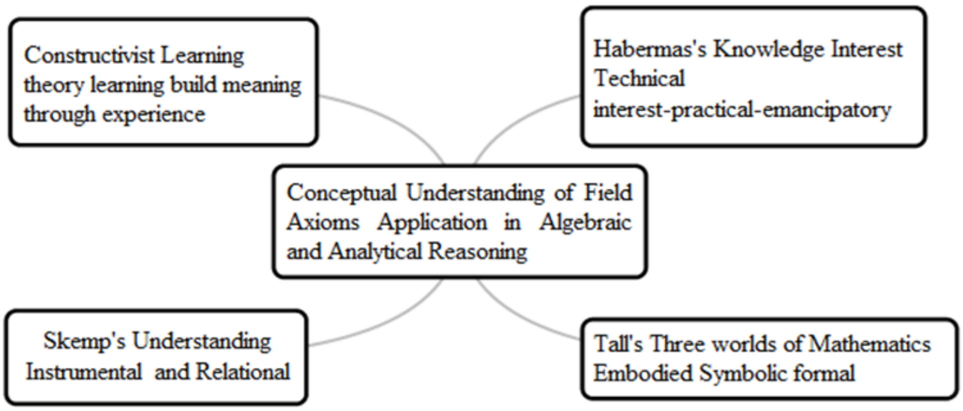
Tall's (2013) three worlds of mathematics, embodied, symbolic, and formal, situate this progression within cognitive development. Understanding the field axioms thus involves bridging embodied and symbolic experiences to formal reasoning. Effective instruction should therefore guide students' movement across these worlds through reflection, discussion, and justification.

Sfard's (2008) communicative theory complements these models by emphasizing the role of discourse and interaction in constructing mathematical meaning. Mathematical learning is viewed as participation in discourse, where thinking develops through communication. Understanding the field axioms, particularly distributivity, requires learners to articulate reasoning, justify transformations, and adopt the linguistic norms of mathematical argumentation. Dialogue, explanation, and collaborative reflection become central mechanisms through which conceptual understanding is internalized.

Together, these theories form an integrative framework describing students' conceptual development of field axioms as a dynamic process of construction, communication, and reflection. Learners begin by constructing meaning through experiences (constructivism), develop relational understanding (Skemp, 1976), progress through reflective knowledge interests (Habermas,1971), traverse embodied–symbolic, formal stages (Tall), and internalize meaning through discourse (Sfard, 2008). This integrative model positions abstraction not as an endpoint but as an iterative cycle where cognitive and communicative processes reinforce one another.

Figure 1

Integrated Theoretical Framework for Students' Understanding of Field Axioms



The theoretical framework plays a central role in guiding this study. First, it helps the researcher interpret student reasoning by categorizing their thinking as instrumental, relational, or emancipatory, based on their speech and behavior. Second, it shapes the classroom observation process by highlighting key moments when students shift between physical, symbolic, and formal types of reasoning. Third, it informs the thematic analysis, ensuring that the study captures both how students build knowledge and how they communicate their understanding. By integrating these elements, the framework supports a deeper exploration of how students learn field axioms in real classroom settings and identifies teaching strategies that best promote meaningful conceptual change.

Method

This study adopted a qualitative interpretive approach grounded in a constructivist paradigm, emphasizing learners' active engagement in constructing meaning through interaction and reflection. Constructivist inquiry seeks to understand how individuals interpret experiences and make sense of phenomena within their learning contexts (Creswell & Poth, 2018). The purpose was to explore how Grade 8 students in Nepal understand and apply the field axioms, particularly the distributive law, during classroom instruction.

A phenomenological design was employed to capture students' lived experiences as they engaged with abstract mathematical ideas. Phenomenology was selected because it enables exploration of how individuals perceive and internalize a phenomenon within their natural context (Moustakas, 1994). The phenomenon under investigation was students' understanding and application of field axioms, particularly distributivity, as they transitioned from arithmetic to algebraic reasoning. The design facilitated an in-depth interpretation of not only what students knew but also how they reasoned, justified, and verbalized their understanding of mathematical relationships.

The study was conducted in one private school located in Baglung Municipality, Baglung District, Nepal. The mathematics instruction process follows the national curriculum developed by the Curriculum Development Centre (CDC, 2021). Instructional practices in this context often emphasize procedural mastery and rule memorization, creating limited opportunities for students to explore underlying logical structures (Bhattarai & Basnet, 2022). The researcher, an experienced mathematics educator, taught one Grade 8 class for a continuous period of fifteen instructional days, focusing on the *algebra* unit that covered factorization and LCM–HCF. Instruction was planned to encourage conceptual linking of field axioms through guided questioning and participatory activities rather than rote demonstration. Out of 30 students in this class, a random drawing was held to select 15 students as the focal participants. The group consisted of 8 girls and seven boys, aged 11-13, with a range of achievement levels and learning dispositions. The names of Bikash, Sita, Ramesh, Kabita, and Nisha were given to the students for the sake of confidentiality. The school management, parents, and students gave their informed consent for participation.

Data were collected through classroom observation, task-based interviews, and Field notes, providing methodological triangulation and enhancing the credibility of interpretation. During classroom instruction, the researcher acted as a participant observer, recording detailed field notes on classroom discourse, gestures, and students' reasoning patterns as they worked through distributive and associative operations. Observations focused on moments of conceptual engagement and instances of confusion or procedural reliance. Following the initial instructional sessions, semi-structured interviews were conducted individually with each participant. Students were asked to solve problems such as *3(x + 2)*, *ab + ac = a(b + c)*, and *a − (b + c)*, and to explain their reasoning. Probing questions like "Why does this rule work?" and "Could it be done differently?" were used to elicit reflective thinking. Each interview lasted about 25 minutes and was recorded with prior consent. To complement these data, students' notebooks, worksheets, and a monthly test were examined to identify how they symbolically represented distributive relationships and whether their written reasoning indicated rule-following or conceptual understanding. These three data sources together enabled a comprehensive view of how students internalized and applied field axioms.

The analysis followed Braun and Clarke's (2006) six-phase thematic analysis framework: familiarization, coding, theme construction, review, definition, and reporting. Transcripts from interviews and observation notes were read repeatedly and coded manually. Codes such as *rule-following*, *partial justification*, *symbolic linkage*, and *reflective reasoning* were developed to capture variations in students' understanding—the study's theoretical framework guided interpretation. Instances of unreflective rule application were categorized as instrumental understanding (Skemp, 1976), while evidence of sense-making or reasoning within context reflected practical reasoning (Habermas, 1971). When students generalized or reflected on the logical necessity of operations, their reasoning was classified as emancipatory understanding. Patterns were further analyzed in relation to Tall's (2013) model of embodied, symbolic, and formal reasoning, enabling examination of how students progressed toward abstraction. A thematic matrix was prepared to trace each participant's movement across levels of understanding based on triangulated evidence from classroom discourse, interviews, and written work. This analytic integration provided a nuanced account of how students in algebra learning cognitively and linguistically internalize field axioms.

The credibility of the findings was improved by triangulation (combination of different data sources) and member checking to ensure that the findings reflected common viewpoints. Rigor was achieved with an audit trail of coding decisions and reflective notes, as well as peer debriefing by two mathematics-education experts during the interpretation of themes. Content validity was ensured through a comprehensive contextual and methodological description, enabling replication in similar educational settings.

Finding and Discussion

The analysis of classroom observations, interviews, and students' written work revealed five interrelated themes that describe how Grade 8 students conceptualized and applied field axioms in algebra. These themes, subthemes, and representative codes are summarized in Table 1, which guided the interpretation presented in the following sections.

Procedural Dominance and Surface Understanding

The majority of the students (8 of 15) demonstrated procedural fluency but were poor in in-depth understanding when applying the distributive law, as shown in Table 2. For example, when asked to explain 4(x+5), Rina quickly responded with 4x+20 but could not explain why it works, simply replying, "Sir, I think it is multiplying by everything." Similarly, Milan described distributivity as a rule for operation.

Table 1

Themes, Subthemes, and Codes on Students' Understanding of Field Axioms

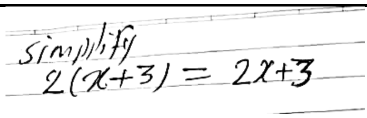
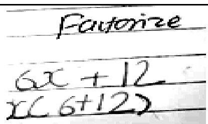
Themes	Subthemes	Illustrative Codes	Conceptual Interpretation
Procedural Dominance and Surface Understanding	Rule memorization without conceptualization; reliance on teacher demonstration; rapid procedural recall	"Multiply everything inside"; formula rule, rote steps, no reasoning	Reflects instrumental understanding (Skemp, 1976) and dominance of technical interest (Habermas, 1971); learning is limited to mechanical application.
Confusion Between Distribution and Expansion	Difficulty in reversing distributive reasoning; viewing factorization as division; empirical rather than logical verification	x (6+12) error; divides both terms; checking by substitution	Shows unidirectional reasoning; lack of awareness that distributivity is a bidirectional property within the field structure (Linchevski, 1995).
Weak Connection to Field Structure and Axiomatic Reasoning	Absence of explicit reference to field properties; compartmentalization of arithmetic and algebra; intuitive rather than deductive thinking	"It happens always"; calculate; no idea of closure/commutativity	Indicates limited movement toward the formal world (Tall, 2013) and a lack of discourse reflecting axiomatic logic (Artigue, 2013; Sfard, 2008).
Instructional Influence and Classroom Culture	Teacher-centered pedagogy; focus on accuracy and speed; minimal dialogic interaction	"Teacher says it is a rule from a formula"; no why questions, fixed exercise pattern.	Represents the dominance of procedural culture; classroom norms restrict communicative abstraction (Sfard, 2008) and practical knowledge interest (Habermas, 1971).
Emergent Conceptual Shifts During Reflection	Use of visual reasoning (area models), contextualized explanations, and reflective realization of reversibility	"Area of rectangle = ab + ac"; "joining rectangles"; conceptual awakening	Demonstrates transition from embodied → symbolic → formal worlds (Tall, 2013); emergence of relational understanding (Skemp, 1976) and reflective abstraction (von Glasersfeld, 1995).

Field notes from class activities revealed that students prioritized speed and accuracy. During drills, the teacher (researcher) repeatedly instructed, "Multiply the number by each term inside the bracket," emphasizing procedural reinforcement over reasoning. In follow-up interviews, many students paused when asked to explain *why* the process worked; several admitted, "We just follow the rule." Analysis of worksheets showed correct but repetitive expansions, often missing explanatory notes or algebraic justification.

Table 2

Authentic-Style Students' Responses from Answer Sheets and Interviews Reflecting Each Theme

Theme	Sample Student Responses (as written on paper or said in interview)	Interpretive Comment

Procedural Dominance and Surface Understanding	 • *"We should multiply both number and letter."* • *2(x + 3) = 2x + 3* written with no correction. • *"I think we should always open a bracket like this."*	These answers show mechanical application without reflection. Students follow a remembered pattern, not reasoning from the property itself. Their explanation depends on authority ("Sir told us"), indicating rule imitation.
Confusion Between Distribution and Expansion	 *"6x + 12 = x(6 + 12)"* written and marked as correct by student. • *"I divide by x because both have x."* • *"To check I put x = 2 and both side same."*	Learners confuse factorization with division, seeing distribution as a one-way process. Verification by substitution shows empirical, not logical, justification.
Weak Connection to Field Structure and Axiomatic Reasoning	• *"It always works, I do not know why."* • *"Because add then multiply is the same."* • *In the answer sheet: 'formula rule' written beside a (b + c).*	Students express belief-based reasoning and lack awareness of the field properties (closure, commutativity, identity). Their statements mix intuition with everyday reasoning, not axiomatic thinking.
Instructional Influence and Classroom Culture	• *"Teacher said do not ask why, just remember the formula."* • *"We learn from example, then copy the same in the exam."* • *Notebook page shows rows of similar problems without explanation lines.*	Reflects the authoritative and performance-oriented classroom. Students associate correctness with copying structure, not exploration. The culture rewards accuracy over justification.
Emergent Conceptual Shifts During Reflection	• *"Area of long box is ab + ac, so that is why we open the bracket."* (student drew two rectangles) • *"Taking common means making small again."* • *"Same work backward, both true."* • Notebook diagram: rectangle divided into two parts labeled *ab* and *ac*.	Students begin connecting procedural steps to visual or contextual meaning. Their use of area analogy and "backward" reasoning indicates developing relational understanding and reflective abstraction.

These patterns indicate a clear dominance of procedural knowledge, where learners execute steps correctly yet fail to articulate underlying concepts, showing what Skemp (1976) termed *instrumental understanding*: knowing how without knowing why.

From a theoretical standpoint, this aligns with Skemp's (1976) distinction between instrumental and relational understanding. Within Habermas's (1971) framework, their reasoning represents the technical knowledge interest, focused on correct task execution rather than on interpretive or emancipatory reflection.

Recent empirical studies (e.g., Bhattrai and Besnet, 2022) show that procedural dominance is widespread in algebra learning. These tendencies are evident in Nepal's instructional context, where assessment systems emphasize correctness and speed. From a constructivist perspective (Von Glasersfeld, 1995), the students' knowledge construction was limited by a lack of metacognitive engagement. They reproduced patterns modeled by the teacher but did not reconstruct or test meaning through reflection or dialogue. This limitation suggests an underdeveloped practical knowledge interest (Habermas, 1971), which mediates the shift from technique to interpretation.

Further, the lack of dialogic classroom discourse restricted opportunities for what Sfard (2008) terms commognitive shifts: transitions from using procedural narratives ("multiply both") to structural ones ("because multiplication distributes over addition"). Research after 2020 highlights how discourse-rich environments facilitate such shifts. For instance, Lu et al. (2022) demonstrated that purposeful teacher questioning, asking why a rule works, enables learners to reformulate operational statements into structural reasoning. Similarly, Mastuti et al. (2022) found that dialogic teaching grounded in the commognitive approach significantly improved reflective reasoning among Grade 8 students.

The procedural dominance observed here also aligns with global findings emphasizing the need for conceptual-procedural integration. While procedural fluency is necessary, its value depends on being intertwined with an understanding of relationships and properties (National Research Council, 2001). Instructional reforms should therefore merge doing and understanding rather than treating them as sequential stages. Pedagogically, these results suggest several strategies. First, lessons should include bidirectional reasoning tasks, for instance, asking students to both expand and factor expressions to illustrate the logical reversibility of distributivity. Second, teachers should employ embodied representations, such as area models or algebra tiles, to help learners visualize how distributive relationships operate. Such visual tools help transition from embodied to symbolic reasoning, facilitating conceptual bridging (Margulieux, Catrambone, & Guzdial, 2021). Third, integrating reflective prompts, for example, "Which property justifies your step?"- can cultivate metacognitive awareness, encouraging movement toward relational and emancipatory reasoning.

The theme of Procedural Dominance and Surface Understanding exposes a fundamental tension in algebra instruction: fluency without reflection. Students' competence in mechanical expansion conceals fragility in the conceptual foundation. Without structured reflection and communicative justification, their knowledge remains instrumentally effective for repetition, yet brittle in unfamiliar contexts. Addressing this imbalance requires instructional designs that merge procedural practice with reflective discourse, guiding learners from the technical to the practical and ultimately to the emancipatory levels of understanding (Habermas, 1971).

Confusion Between Distribution and Expansion

Students exhibited persistent confusion between *distribution* and *expansion*, particularly when asked to reverse distributive reasoning during factorization. Classroom and worksheet analyses revealed that learners treated the distributive property a(b+c) =ab+ac as a one-directional rule rather than a reversible law. For instance, when simplifying 6x+12, Bimal wrote x (6+12) and justified, "I take out xxx because it is common." Laxmi successfully factorized 8y+12=4(2y+3) but explained, "Four divides both," expressing an arithmetic rather than algebraic understanding.

In a follow-up task (Major result shown in Table 2), Sabin tested 4(x+3) and 4x+12 by substituting x=2, then commented, "May be it always works!", an indication of discovery through empirical trial rather than through logical justification. The field notes confirmed that most students were comfortable expanding expressions (e.g., 3(x+2) but struggled when required to compress or factorize them. Their written work frequently contained crossed arrows or comments such as *"take out"* or *"divide both"*, revealing procedural recall rather than conceptual comprehension.

This evidence shows that the students did not recognize distributivity as a *bidirectional relationship* connecting multiplication and addition within the field axioms of real numbers. The confusion between distribution and expansion highlights a limited structural understanding of algebraic operations. Learners perceived the distributive law as a rule for expansion rather than as a property of equivalence that applies symmetrically to both distribution and factorization. This pattern aligns with Skemp's (1976) notion of instrumental understanding, knowing how to perform a rule without comprehending its rationale. Consistent with Habermas's (1971) framework of knowledge-constitutive interests, students operated within a technical interest, applying procedures efficiently but without reflection. They had not developed the practical interest of interpretive reasoning or the emancipatory interest of critical abstraction. Their reasoning often substituted informal expressions such as "divide both" or "take out" for formal explanation, showing reliance on everyday arithmetic rather than algebraic structure.

From a constructivist standpoint, this confusion represents a failure of reconstruction. As von Glasersfeld (1995) explained, learning involves reorganizing prior knowledge into new conceptual frameworks. The tendencies appeared here, where learners' procedural competence masked their lack of structural integration. The language used, phrases like "take out" or "multiply both," also supports Sfard's (2008) communicative theory, which views learning as a shift in discourse. Students' utterances reflected ritualized communication, aimed at reproducing correct forms rather than constructing justification or relational meaning.

From a cognitive-developmental lens, learners appeared confined to Tall's (2013) symbolic world, manipulating symbols without reference to the underlying axiomatic structure of real numbers. Without guided reflection, they could not progress to the formal world, where distributivity is understood as a logical law linking multiplication and addition.

In the Nepali context, this confusion mirrors classroom practices that emphasize procedural fluency and examination outcomes over reflective reasoning (Bhattarai & Basnet, 2022). Students rarely encounter tasks requiring them to verify equivalence or reason bidirectionally. Such pedagogical habits promote what Akhtar et al. (2020) called symbolic imitation, where students replicate patterns without understanding their logic. To cultivate relational understanding, teachers should integrate reflection and justification within instruction. Bidirectional reasoning tasks, such as expanding and refactoring the same expression, can help learners perceive reversibility. Visual tools, including area models, may bridge Tall's embodied and symbolic worlds, promoting reflective abstraction. Structured discussion prompts "What property allows this step?" or "Can you show it works backward?" can further encourage shifts from technical execution to emancipatory reasoning (Habermas, 1971). Hence, Students' confusion between distribution and expansion stems from rule-based teaching and can be addressed by fostering reflective reasoning and visual understanding of distributivity as a logical]r35elationship.

Weak Connection to Field Structure and Axiomatic Reasoning

Most students failed to connect algebraic rules to the field axioms underlying the real number system. When asked why a(b+c)=ab+ac, Sita replied, "Because it always happens," while Dipesh said, "We just calculate; we do not think of rules like field or closure." None of the participants referred to commutativity, associativity, or closure as justifications for their reasoning. Their answers showed that distributive relationships were understood as *familiar rules* rather than as *axiomatic properties* defining number structure.

Classroom observation revealed that learners viewed axioms as remote, textbook ideas rather than living principles governing mathematical reasoning. During exercises, when the teacher (researcher) prompted, "What property allows this step?", students typically paused or repeated, "It is the formula." In their written work, algebraic manipulations appeared correct, yet no evidence of justification or mention of properties such as commutativity or identity was present. This pattern suggests that students compartmentalized arithmetic and algebra: they could perform symbolic procedures but did not perceive these as manifestations of the field structure of real numbers. Their reasoning was intuitive and empirical, not deductive or axiomatic.

This study reveals a significant disconnect between students' procedural fluency and their structural reasoning in algebra. Learners often execute operations without recognizing their foundation in the axioms of the real-number field. Explanations such as "it always happens" reflect belief-based reasoning rather than logical necessity, indicating a technical epistemological orientation (Habermas, 1971) focused on accuracy over conceptual understanding.

From a constructivist perspective, this gap signifies a failure to reconstruct procedural knowledge into an abstract, axiomatic framework (von Glasersfeld, 1995). Students demonstrated instrumental understanding (Skemp, 1976), knowing how to perform operations but not why they hold. Recent studies affirm that limited attention to field properties undermines conceptual depth. Hunter et al. (2022) found that students treat algebraic rules as isolated algorithms. Tall's (2013) cognitive model situates these learners within the symbolic world, where manipulation of symbols occurs without understanding their formal meaning. Their reasoning relied on pattern recognition rather than logical generalization. Transitioning to the formal world requires instruction that explicitly connects procedural steps to axiomatic justification. Sfard's (2008) commognitive framework further clarifies this disconnect: students' discourse lacked formal mathematical language, favoring colloquial expressions over terms like "distributivity" or "closure." This linguistic gap suggests exclusion from mathematical discourse communities. Studies by Lu et al. (2022) and Mastuti et al. (2022) show that modeling formal language and dialogic teaching enhances structural reasoning and metacognitive awareness. Empirical evidence supports these findings. Thus, deep algebraic understanding arises only when learners perceive operations as reflections of an internally consistent axiomatic system.

Activities such as axiom mapping, proof-based discussion, and visual models (e.g., area representations) can foster transitions from procedural to structural understanding (Ünal et al., 2023; Margulieux et al., 2021). In Nepal, this disconnect reflects systemic instructional norms. Bhattarai & Basnet (2022) noted that axioms are taught as abstract definitions, detached from reasoning. The current study confirms that even proficient students cannot justify operations. Curriculum reform must reframe axioms as epistemic foundations guiding mathematical reasoning. Finally, bridging procedural fluency with axiomatic awareness is essential for cultivating relational and emancipatory understanding in mathematics.

Instructional Influence and Classroom Culture

The classroom culture and instructional practices strongly influenced how students conceptualized mathematical rules. Observations revealed a predominantly teacher-centered environment emphasizing accuracy, speed, and obedience over reasoning and discussion. During lessons, the teacher (researcher) observed that students often relied on procedural instructions rather than exploring alternative solutions. When Rojina was asked why she applied a specific rule, she responded, “If we ask why, the teacher says it is a rule from the formula.” Similarly, Kritika commented, “We learn the steps, but if the question changes, we become confused.” Lesson routines followed a fixed pattern: teacher demonstration, collective repetition, and written practice. Few opportunities existed for open questioning or reflective dialogue. Students’ notebooks revealed minimal written explanations beyond symbolic steps. These observations reflected a performance-oriented culture, where mathematical success was measured by correctness and completion rather than comprehension. The findings suggest that the classroom culture and instructional design reinforced procedural dependence. Learners perceived mathematical rules as authority-based directives, not as conceptual relationships to explore or justify.

This theme underscores how instructional culture mediates students’ epistemological orientation toward mathematics. In classrooms dominated by procedural and authoritative teaching, knowledge is positioned as externally transmitted rather than internally constructed. Within Habermas’s (1971) framework, such environments reflect a technical knowledge interest, emphasizing task efficiency over conceptual understanding or critical reflection.

Constructivist theory highlights that these settings restrict opportunities for self-regulated knowledge construction (von Glasersfeld, 1995). Students replicate demonstrated steps without reconstructing meaning or testing conceptual coherence, resulting in instrumental understanding (Skemp, 1976) devoid of relational depth. Empirical studies reinforce this interpretation. Bhattarai and Basnet (2022 described Nepali classrooms as teacher-centered, prioritizing procedural correctness and exa1m preparation at the expense of conceptual reasoning. Akhtar et al. (2020) similarly observed that South Asian students often engage in “pattern imitation,” memorizing rules without justification. This cultural pattern was evident in the current study, where students equated learning with formulaic obedience.

Global research post-2020 confirms that instructional culture profoundly shapes reasoning. The classrooms fostering student agency and discussion promote deeper structural understanding, while those emphasizing correct answers encourage surface learning. Chan (2023) reported that non-dialogic environments hinder conceptual transfer despite procedural proficiency. Mastuti et al. (2022), using Sfard’s (2008) communicative framework, showed that dialogic teaching, where students verbalize and challenge reasoning, enhances metacognitive awareness and relational thinking.

In this study, classroom discourse was ritualized rather than exploratory. Statements like “Sir said to multiply” reflect ritual participation rather than object-level communication. As Lu et al. (2022) argue, teacher prompts that demand justification are essential for shifting discourse toward conceptual engagement. Cognitively, the absence of reflective dialogue confines students to Tall’s (2013) symbolic world, where expressions are manipulated without connection to embodied or formal reasoning. Without guided transitions across these cognitive domains, students struggle to generalize properties or perceive axioms as structurally interrelated. Design-based interventions offer promising alternatives. Davis and Ross (2024) demonstrated that “reasoning pauses” during instruction enhance structural retention. Sibgatullin et al. (2022) found that prompting explanation before execution fosters relational understanding. These studies affirm that instructional culture is not neutral; it either reinforces or disrupts procedural dominance.

In Nepal, reform is urgently needed. Teacher training should prioritize dialogic interaction, reasoning-based questioning, and collective reflection. Manandhar et al. (2022) emphasize that algebraic understanding deepens when learners justify transformations and explore relationships verbally. Such practices align with Habermas's emancipatory interest, guiding students toward autonomous reasoning. In conclusion, instructional culture defines whether mathematical understanding stagnates or evolves. Transforming classroom norms is central, not supplementary, to cultivating relational and axiomatic reasoning.

Emergent Conceptual Shifts During Reflection

Despite the dominance of procedural instruction, several students demonstrated initial conc1eptual transformation when given opportunities to reflect, visualize, and explain their reasoning. During interviews and guided activities using area models, a few learners moved beyond memorized procedures toward meaningful reasoning about the distributive law. For example, Bikash explained, "If a rectangle has sides: a and (b+c), its area is ab+ac. That is why we multiply." Similarly, Laxmi elaborated, "Because we can join two rectangles to make one big one," indicating the use of visualization to connect arithmetic and algebraic representations. Another student, Nisha, noted during reflection, "When we take common or open brackets, we are doing the same work backward." These remarks contrasted with earlier responses focused on rules such as "multiply everything inside." Their written work also evolved; later worksheets contained annotations like *"same area → distributivity"* and arrows showing equivalence between expanded and factored forms. This progression suggests that conceptual understanding can begin to emerge when instruction provides space for reflection and visual reasoning.

This theme highlights the transformative role of reflection in shifting students from instrumental to relational understanding. When prompted to reason visually or contextually, learners began linking procedural actions to underlying mathematical structures. As Skemp (1976) noted, this transition—from knowing how to knowing why marks the foundation of relational understanding. Within Habermas's (1971) framework of knowledge interests, students' reasoning evolved from a technical orientation, focused on procedural accuracy, toward a practical interest, where meaning and justification became central. Reflections on reversibility ("same work backward") further suggest movement toward an emancipatory interest, characterized by metacognitive awareness and self-directed reasoning. Cognitively, these developments align with Tall's (2013) model of mathematical thinking, where learners progress from embodied experiences to symbolic manipulation and ultimately to formal reasoning. Visual explanations by students like Bikash and Laxmi illustrate the embodied stage, where geometric reasoning anchors abstraction. As they verbalized these relationships, they began bridging embodied and symbolic worlds through "reflective compression," the reorganization of prior knowledge into new structures.

This process exemplifies von Glasersfeld's (1995) concept of reflective abstraction, wherein learners reconstruct meaning through awareness of their mental operations. Recognizing distributivity as both expansion and factorization reflects early structural generalization. Empirical studies support these interpretations. Ünal et al. (2023) found that visual models, particularly area representations, enhance understanding of distributivity and support transitions from procedural to conceptual reasoning. Margulieux et al. (2021) showed that contrasting cases and reflective prompts help integrate multiple conceptions, fostering a more profound understanding.

Umer and Mwanza (2023) demonstrated that dialogic reflection, where students articulate reasoning aloud, improves conceptual coherence and metacognitive control. Similarly, Chan (2023) confirmed that conceptual understanding, rather than procedural speed, predicts long-term retention and transfer in algebra. From a commognitive perspective (Sfard, 2008), these reflective episodes represent discourse transformation. Students' ritualized phrases ("multiply both") evolved into exploratory narratives ("we join rectangles"), signaling participation in mathematical discourse. Lu et al. (2022) emphasized that such linguistic shifts indicate genuine conceptual change. Pedagogically, these findings underscore the power of reflection and visualization. Strategies like reasoning pauses (Schoenherr et al., 2024), embodied tools (e.g., area models, algebra tiles), and reflective questioning ("Can you show it works both ways?") can bridge procedural fluency and structural reasoning.

The emergence of relational understanding in this study affirms that conceptual change is possible within procedural environments when reflection, visualization, and dialogue are embedded. Ultimately, this theme reveals the beginnings of emancipatory learning: students recognizing mathematical rules as logical necessities, not arbitrary prescriptions.

Conclusion

This study revealed that Grade 8 students in Nepal predominantly exhibit procedural dominance and surface understanding of the distributive law, with persistent confusion between distribution and expansion and a weak connection to field axioms. However, moments of reflective engagement demonstrated the potential for conceptual transformation when instruction incorporated dialogue, visualization, and reasoning. The findings underscore that classroom culture anchored in procedural authority remains a critical barrier to structural understanding. The broader implication is that m1athematics education must shift from rule transmission to reflective discourse, enabling learners to internalize axiomatic reasoning as part of their mathematical identity. The study contributes to the growing body of research connecting constructivist, commognitive, and axiomatic perspectives in developing algebraic understanding. Future research should investigate how teacher professional development and curricular design can support reflective practices and promote relational understanding across diverse contexts. As a call to action, mathematics educators are urged to cultivate classrooms that value justification over memorization, empowering students to see mathematics not as a set of procedures but as a coherent, logical system for reasoning and reflection.

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