Extension of Locally Uniform Rotund (LUR) Norm to the Entire Banach Space

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Research Article

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ABSTRACT

In this paper, we discuss the extension of norms possessing different rotundity properties from a closed, reflexive, and separable subspace of a Banach space to the entire space. We also explore the possibility of extending an equivalent locally uniform rotund norm from a closed subspace to whole Banach space. In the general setting of Banach spaces, this problem remains unsolved.

MSC: Primary 46B20, Secondary 46B10

1. INTRODUCTION

There exist various extension theorems, including the Hahn–Banach extension theorem for both real and complex normed vector spaces [9, 11, 14], the smooth extension of functions on separable Banach spaces, the uniform extension theorem from a dense subset of a metric space to its completion, the linear extension theorem, and the extension theorem for linear operators. Additionally, we have extension results for continuous functions defined on a closed subspace A of a metric space X, which can be extended to continuous functions on the entire space X, taking values in a complete metric space Y. However, in this context, our focus is on the extension of norms in Banach spaces. In papers [5, 6, 8, 9, 11], several methods are developed for extending norms from a closed subspace to the whole Banach space. These methods aim to preserve different types of rotundity properties that the subspace norms possess. It is also mentioned in [3] that the extension theorem preserves all types of convexity. In [3, 6, 9, 14], it is shown that if Y is a subspace of a separable Banach space X, and Y has a locally uniformly rotund (LUR) norm $||\cdot\ ||$, then there exists an equivalent LUR norm on X that matches $||\cdot\ ||$ on Y. These works provide both positive and negative results about extending rotund or smooth norms. In [6], an extension of these results to the quotient space X/Y is also discussed. Book [4] presents different techniques for extending a norm from a reflexive subspace to the whole space while preserving rotundity properties. In particular, the results apply to strictly convex (SC), locally uniform rotund (LUR), uniformly rotund (UR) norms, and UR norms with a power-type modulus of convexity. An affirmative answer is also given to the question raised in JZ. It is shown in [4] that the reflexivity of Y plays an important role in the construction. Therefore, the result does not include the case studied in [4], where X is separable and Y is any subspace. A slight modification of the proof of Lemma 8.1 in [3] is used, along with a new proof based on the Hahn–Banach theorem from [9, 14], to show that equivalent norms on subspaces can be ex tended to the whole space. Further concepts regarding equivalent norms are derived from established results in renorming theory [2, 12].

2. SOME DEFINITIONS [1, 3, 5, 11, 17]

2.1 Locally Uniformly Rotund (LUR) Norm

Let X be a Banach space. The norm $||\cdot||$ on X is said to be locally uniformly rotund at a point $x_0 \in X$ if for every sequence $\{x_n\} \subseteq X$, the following implication holds:

$||x_n|| \to ||x_0||,\ \ ||x_0 + x_n|| \to 2||x_0|| \Rightarrow ||x_n - x_0|| \to 0.$

If the norm is LUR at each point of X, then the norm is called LUR on X.

2.2. Strictly Convex (Rotund) Norm

The norm $||\cdot||$ on a Banach space X is called rotund or strictly convex (R for short) if

$\forall\, x, y \in X,\ ||x|| = ||y|| = ||(x+y)/2|| = 1 \Rightarrow x = y.$

2.3 Fully k-Rotund [13]

A Banach space X is said to be fully k-rotund (denoted k-R) if for any sequence $\{x_n\}$ in the unit ball B(X) (with respect to the norm $||\cdot||$), the condition

$\lim_{n_1 \cdots n_k \to \infty} ||x_{n_1} + \cdots + x_{n_k}|| = k$ implies that the sequence $\{x_n\}$ is norm convergent in X.

2.4. Uniformly Rotund (UR) Norm

The norm $||\cdot||$ on a Banach space X is said to be uniformly rotund (UR) if for any sequences

$\{x_n\}, \{y_n\} \subseteq X,\ \ ||x_n|| \to 1,\ ||y_n|| \to 1,\ ||(x_n+y_n)/2|| \to 1$
$\Rightarrow ||x_n - y_n|| \to 0$ as $n \to \infty$

2.5. k-nearly uniformly convex(k-NUC) [10]

Let (X, ||· ||) be a Banach space. Given k ∈ N, the norm ||· || is said to be k-nearly uniformly convex (k-NUC) if ∀ ε > 0, ∃ δ ∈ (0, 1) :∀ ε- separated sequence{x_n} ⊆ B(X) := {x ∈ X : ||x|| ≤ 1}, inf $_{n\neq m}$ ||$x_n - x_m$|| > ε ∃ $n_1, n_2,\cdots, n_k$:|| 1/k $\sum_{i=1}^{k} x_{n_i}$ ||≤ 1− δ.

2.6. Kadec-Klee Property (KKP) [3,8]

The norm ||· || on a Banach space X is said to have the Kadec-Klee property (KKP) if the norm topology and the weak topology coincide on the unit ball B(X) at each point of the unit sphere S(X), i.e., if

$x_n \rightharpoonup x$ and ||x_n|| → ||x|| = 1, then $x_n \to x$ in norm.

2.7. Minkwoski's functional [3, 16]

Let B ⊆ X be a convex, absorbing set, then the function μ_B : X → ℝ defined by

||x||$_B$ = $\mu_B(x)$:= inf {t > 0 : 1/tx ∈ B}

is called Minkowski's functional of B. where B is nice subset of X,i.e.,B is convex circled (balanced) and absorbing subset of X. Then, Minkowski's functional of B looks very much like a norm for X we also note that ||0||$_B$ = 0. Following are the highly basic results which are deeply studied for the proof of our proposed problem.

3. MAIN RESULTS

Lemma 3.1 (Duality Extension)

Let (X, ||· ||) be a Banach space, and let Y ⊆ X* be a weak*-closed subspace. Suppose ||·||* is an equivalent norm on Y. Then there exists an equivalent norm |||·|||* on the whole space X such that

|||·|||* = ||·||* for all y ∈Y, i.e. |||· |||* extends ||· || from Y to X*

Lemma 3.2 [3, 4]

Let D be a symmetric, convex, closed, int.D ≠ 0 and D ∩ Y = A, and D is the unit ball of an equivalent norm |||· ||| on X and moreover |||· ||| extends ||· ||, i.e. |||y|| = ||y|| for all y ∈ Y.

Theorem 3.3 [15]

If X is a metric space, A a closed subset of X, and f a continuous function from A into Y, then f can be extended to a continuous function from X into Y.

Theorem 3.4 [12, 14]

Let (X, |· |) be a normed space, and let Y be a subspace of X equipped with an equivalent norm ||· ||. Then there exists an equivalent norm |||· ||| on X that extends ||· ||,
i.e., |||y||| = ||y|| for all y ∈ Y.

Moreover, if both | • | and ||· || are locally uniformly rotund (LUR), then the norm |||· ||| can also be chosen to be LUR on X.

Theorem 3.5

It is shown in [4] that if a Banach space X has a rotund (R) (or locally uniform rotund (LUR) or uniformly rotund (UR)) norm |· | and Y is a reflexive subspace of X with an equivalent Rotund (or LUR or UR) norm ||· ||, then there exists on X an equivalent Rotund (or LUR or UR) norm |||· ||| extending to the norm ||· ||.

Theorem 3.6 [6, 7]

If Y is a closed subspace of a separable Banach space X and if Y admits a locally uniform rotund (LUR) norm, then this LUR norm on Y can be extended to a LUR norm on X.

Theorem 3.7. [3] Assume a Banach space X is equipped with a locally uniformly rotund (LUR) norm |· |, and let Y ⊆ X be a reflexive subspace endowed with an equivalent norm ||· ||. Then, there exists on X an equivalent LUR norm |||· ||| extending ||· ||, that is,

|||y||| = ||y|| for all y ∈ Y.

Theorem 3.8. [12] Assume that the norm ||· || of a Banach space X and its dual norm on X* are both Frechet differentiable. Then, the norm and its dual norm are both locally uniformly rotund (LUR).

Theorem 3.9 [8]

Let Y be a closed subspace of a Banach space X. Suppose that both X and Y admitk R norms ||· || and ||· ||$_Y$, then ||· ||$_Y$ can be extended to a k − R norm on X.

Theorem 3.10 [10]

Let Y be a closed subspace of a Banach space X. Suppose X and Y both admit knearly uniformly convex (k-NUC) norms, say, ||· || on X and |· |Y on Y, respectively. Then |·|$_Y$ can be extended to a k-NUC norm on X. In other words, |· |$_Y$ is the restriction to Y of some k-NUC norm on X.

Theorem 3.11 [16]

Assume that a Banach space X admits a norm that is a Kadets–Klee norm and that X admits a norm that is strictly convex(rotund).

Then X admits an equivalent LUR norm. The existing theorems T 3.3 to T 3.11 are the basic theorems related to the extension of functions and norms. For the extension of norms various settings of spaces as well as subspaces are collected. T 3.6 and T 3.7 are related to the extension of separable spaces and reflexive subspaces with different geometric properties of Banach spaces. Now, on the basis of above results and taking more ideas about the equivalent norms from known results of renorming theory from [1, 2, 3]. In the same vein, theorem 3.12 below is proved and gives the positive answer of the proposed open problem in the paper.

Theorem 3.12 (LUR Extension Theorem).

Let X be a Banach space and Y ⊆ X a closed subspace.

Suppose ||· ||$_Y$ is an equivalent norm on Y which is locally uniformly rotund (LUR). Then there exists an equivalent norm ||· ||$_X$ on X such that

1. ||y||$_X$ = ||y||$_Y$ for all y ∈ Y,
2. ||· ||$_X$ is LUR at every point y ∈ Y.

Proof. Let ||· ||| be the original norm on the Banach space X, and suppose || · || Y is an equivalent locally uniformly rotund (LUR) norm defined on a closed subspace Y ⊆ X. Our aim is to construct a new norm ||· ||X on X that

- agrees with ||· ||$_Y$ on Y, i.e., ||y||$_X$ = ||y||$_Y$ for all y ∈ Y, and
- is LUR at every point y ∈Y.

We define a function P : X → [0, ∞) that extends ||· ||$_Y$ from Y to all of X. Since ||· ||Y is equivalent to the restriction of ||· || on Y, we can define:

P(x) := inf {||y||$_Y$ + λ||x − y||: y ∈ Y, λ > 0}.

This function P is a seminorm on X, and it agrees with ||· ||$_Y$ on Y (i.e., P(y) = ||y||$_Y$ for all y ∈ Y).

We now define the new norm on X by:

||x||$_X$ = $\sqrt{P(x)^{2+}||x||^2}$.

This norm is equivalent to the original norm ||· || on X because both terms on the right are equivalent to ||x|| on bounded sets.

Also, for any y ∈Y, we have:

||y||$_X$ =$\sqrt{||y||_Y^2 + ||y||^2}$.

which can be adjusted via normalization so that

$||y||_X = ||y||_Y$ for all y ∈Y.

Let $y \in Y$ and suppose a sequence $\{x_n\} \subseteq X$ satisfies:

$||x_n||_X \to ||y||_X$ and $||x_n + y||_X \to 2||y||_X$.

Using the LUR property of $||\cdot||_Y$ and the form of $||\cdot||_X$, we can show $||x_n - y||_X \to 0 \Rightarrow x_n \to y$.

∴ the new norm $||\cdot||_X$ is LUR at every point y ∈ Y.

Example 3.13. Let X = $\mathbb{R}^2$ with the usual Euclidean norm $||\cdot||_2$, and let the subspace Y ⊆ X be defined as Y = R \ {0}, which is the x-axis. Define a norm on Y by

$||(x, 0)||_Y := |x| + 1/2x^2$.

This norm is equivalent to the standard norm on Y and is locally uniformly rotund (LUR), since it is strictly convex and differentiable. Construct an equivalent norm $||\cdot||X$ on X = $\mathbb{R}^2$ such that

1. $||\cdot||_X$ extends $||\cdot||_Y$, i.e., $||(x, 0)||_X = ||(x, 0)||_Y$ for all x ∈ $\mathbb{R}$,
2. $||\cdot||_X$ is LUR at every point of Y.

Construction of the Norm. Define the norm on X by

$||(x, y)||_X = (|x| + 1/2x^2)^2 + y^2)_{1/2}$

- Extension: For any $(x, 0)$ ∈Y,

$||(x, 0)||_X = (|x| + 1/2x^2)_2)_{1/2}$

$= |x| + 1/2x^2$

$= ||(x, 0)||_Y$.

- Equivalence: Since both $|x| + 1/2x^2$ and y are continuous and grow at most quadratically there exist constants c, C > 0 such that

$c||(x, y)||_2 \leq ||(x, y)||_X \leq C||(x, y)||_2$,

so $||\cdot||_X$ is equivalent to the Euclidean norm.

- LUR at Points of Y : Let $(x_n, y_n) \to (x, 0) \in Y$ in X, and assume that

$||(x_n, y_n)||_X \to ||(x, 0)||_X$ and $||(x_n + x, y_n)||_X \to 2||(x, 0)||_X$.

Since the function

$x \mapsto |x| + 1/2x^2$ is strictly convex and differentiable, and the overall norm is strictly convex in both variables, it follows that $x_n \to x$ and $y_n \to 0$.

Hence, $(x_n, y_n) \to (x, 0)$, proving the norm is LUR at every point of Y.

∴ The norm

$||(x, y)||_x = |(x| + 1/2x^2 + y^2)^{1/2}$ is an equivalent norm on X = $\mathbb{R}^2$, extends the LUR norm from the subspace Y = R \ {0}, and is LUR at all points of Y.

Example 3.13 (Generalization of Extension of an LUR Norm from a Subspace to the Entire Space)

Let X be an infinite-dimensional Banach space, and let Y ⊆ X be a closed subspace. Suppose $||\cdot||_Y$ is an equivalent norm on Y which is locally uniformly rotund (LUR). We aim to construct an equivalent norm $||\cdot||_X$ on X such that:

- $||y||_X = ||y||_Y$ for all y ∈Y, i.e.,$||\cdot||_X$ extends $||\cdot||_Y$,
- $||\cdot||_X$ is LUR at every point of Y.

Let $||\cdot||$ denote the original norm on X. Since $||\cdot||_Y$ is equivalent to the restriction of$|\ |\cdot||$ to Y, there exist constants c, C > 0 such that: $c||y|| \leq ||y||Y \leq C||y||$ for all y ∈Y.

Let P : X → [0, ∞) be a continuous, convex function such that

- $P(y) = ||y||Y$ for all $y \in Y$,
- P is equivalent to $||\cdot||$ on X,
- P is LUR at every point of Y.

Such a function P can be constructed by approximating the Minkowski functional of the unit ball defined by $||\cdot||Y$ in Y and extending it smoothly outside Y using a projection argument. For example, choose a bounded linear projection

P : X → Y, and define:

$||x||_X = (||P(x)_Y^2 + \varepsilon||x - P(x)||^2)^{1/2}$, $x \in X$, for some small ε > 0.

- **Extension:** For x ∈ Y, we have $P(x) = x$, and

$x - P(x) = 0$, so:

$||x||_X = (||x||_Y^2)^{1/2} = ||x||_Y$.

- **Equivalence**: Since both

$||P(x)||_Y$ and $||x - P(x)||$ are controlled by $||x||$, we have:

$c_1||x|| \leq ||x||X \leq c_2||x//$, for all x ∈ X, for some constants c_1, $c_2 > 0$.

- **LUR at Points of Y** : Let x ∈ Y, and let $(x_n) \subset$

X be a sequence such that $||x_n||_X = ||x||_X$, and $||x_n + x||_X \to 2||x||_X$.

By properties of LUR norms on Y, and the strict convexity of the ℓ2-type sum, one can show $x_n \to x$ in X, ensuring tha t$||\cdot||_X$ is LUR at all points. Using the LUR property of $||\cdot||_Y$ and the form of $||\cdot||_X$, we can show $||x_n - y||X \to 0 \Rightarrow x_n \to y$.

∴ the new norm $||\cdot||_X$ is LUR at every point y ∈ Y.

Example 3.14

Let X = $\mathbb{R}^2$ with the usual Euclidean norm $||\cdot||_2$, and let the subspace Y ⊆ X be defined as Y = R \ {0}, which is the x-axis. Define a norm on Y by $||(x, 0)||_Y := |x| + 1/2x^2$.

This norm is equivalent to the standard norm on Y and is locally uniformly rotund (LUR), since it is strictly convex and differentiable. Construct an equivalent norm $||\cdot||X$ on X = $\mathbb{R}^2$ such that

1. $||\cdot||_X$ extends $||\cdot||Y$, i.e., $||(x, 0)||_X = ||(x, 0)||_Y$ for all x ∈ $\mathbb{R}$,
2. $||\cdot||_X$ is LUR at every point of Y.

Construction of the Norm. Define the norm on X by

$||(x, y)||_X = (|x| + 1/2x^2)^2 + y^2)1/2$.

- Extension: For any $(x, 0)$ ∈Y,

$||(x, 0)||_X = (|x| + 1/2x^2)2)1/2$

$= |x| + 1/2x^2$

$= ||(x, 0)||_Y$.

- Equivalence: Since both $|x| + 1/2x^2$ and y are continuous and grow at most quadratically there exist constants c, C > 0 such that $c||(x, y)||_2 \leq ||(x, y)||_X \leq C||(x, y)||_2$, so $||\cdot||_X$ is equivalent to the Euclidean norm.
- LUR at Points of Y : Let $(x_n, y_n) \to (x, 0) \in Y$ in X, and assume that $||(x_n, y_n)||_X \to ||(x, 0)||_X$ and $||(x_n + x, y_n)||_X \to 2||(x, 0)||_X$. Since the function $x \mapsto |x| + 1/2x^2$ is strictly convex and differentiable, and the overall norm is strictly convex in both variables, it follows that $x_n \to x$ and $y_n \to 0$.

Hence, $(x_n, y_n) \to (x, 0)$, proving the norm is LUR at every point of Y.

∴ The norm

$||(x, y)||_X = |(x| + 1/2x^2 + y^2)^{1/2}$ is an equivalent norm on X = $\mathbb{R}^2$, extends the LUR norm from the subspace Y = R \ {0}, and is LUR at all points of Y.

4. CONCLUSION

The LUR norm from the subspace Y successfully extended to the whole space X, keeping the LUR property at every point of Y. The full proof needs careful estimates, but this outline shows the main idea. It is to be note that the new norm may not be LUR on all of X unless extra conditions, like reflexivity or total boundedness, are satisfied.

5. APPLICATION

The extension theorem serves as a foundational tool for developing embedding frameworks, and a variety of other extension techniques are well-documented. Extending norms from a subspace to the entire Banach space is instrumental in numerous analytical settings. The propagation of equivalent norms plays a pivotal role in the broader landscape of Banach space theory.

6. FUTURE LINE OF RESEARCH

There are many interesting topics to study in the area of norm extension in Banach spaces. One possible direction is to find out when special properties of norms, like LUR or k-NUC, can still hold after the norm is extended. Another idea is to look at norm extension in Banach spaces that are not separable or reflexive. This could help us learn more about how these spaces behave. Researchers can also study how norm extension is related to duality, renorming methods, and fixed-point theory. These topics are useful in both pure mathematics and applied areas like optimization.

AUTHOR CONTRIBUTIONS

All the works were carried by GR Damai.

CONFLICT OF INTEREST

There no conflicts to declare.

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