Confluent Hypergeometric Function and Its Nexus to Different Special Functions

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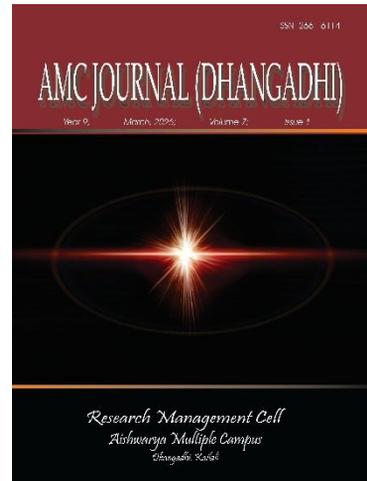
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Abstract

This paper aims to explore the unifying role of the Confluent Hypergeometric Function in the derivation of some special functions: Incomplete Gamma Function, Error Function, Laguerre Polynomial Function, Hermite Polynomial Function, Bessel Function, and Whittaker Function through series expansions. This study is based on a systematic review of the literature to lay out a cohesive perspective that enhances understanding of the theoretical concepts of special functions and furnishes a streamlined approach for applications across mathematics, engineering, probability, and other areas of applied mathematics.

Keywords: Bessel Function, Confluent Hypergeometric Function, Error Function, Incomplete Gamma Function, Hermit Polynomial Functions, Whittaker Functions

Introduction

The solution of differential equations or integrals that cannot be expressed in the form of elementary functions like polynomial functions, exponential functions, trigonometric, and logarithm functions are called the **Special Functions.** Hypergeometric Function, Incomplete Gamma Function, Error Function, Laguerre Polynomial Function, Hermite Polynomial Function, Bessel Function, and Whittaker Function are some of the examples of special functions.

The function

$$_2F_1(p,q;r;x) = \sum_{n=0}^{\infty} \frac{(p)_n (q)_n}{(r)_n} \times \frac{x^n}{n!} \qquad [1.1]$$

where $(p)_n, (q)_n \,\&\, (r)_n$ denote Pochhammer symbols defined by

$$(p)_n = p(p+1)(p+2)(p+3)\ldots\ldots[p+(n-1)] \qquad [1.2]$$

is called a hypergeometric function(Rainville,1960). If q = 1 and r is replaced by q then [1.1] reduces to the form

$$_1F_1\left[p;q;x\right] = \sum_{n=0}^{\infty} \frac{(p)_n}{(q)_n} \times \frac{x^n}{n!} \qquad [1.3]$$

is called the **Confluent Hypergeometric Function**. According to Rainville (1960), the function [1.3] is also known as the Pochhammer- Barnes Confluent Hypergeometric Function (Poudel 2023). This function was initially developed by a German Mathematician Ernst Eduard Kummer, in the 19th century, which is also now known as the solution of Kummer's differential equation:

$$x\frac{d^2w}{dx^2} + (q-x)\frac{dw}{dx} - pw = 0, w = {}_1F_1(p;q;x)$$

Euler(1769), Gauss(1813), Kummer (1837), Riemann(1857) and so on developed the concepts, types and various properties of hypergeometric function and later Tricomi(1947) and Slater(1966) Watson(1944), Abramowitz and Stegun (1964) established numerous relationships among the special functions (Andrews, Askey & Roy (1999), Olver et.al. 2010). These works lacked a framework of the derivation to focus on the unified approach addressing the role of Confluent hypergeometric function at arena of the special functions. A unified approach enhances the understanding of theoretical concepts and streamlines their application in solving differential equations within the broader platform of pure and applied mathematics.

In this study, we aim to explore the nexus of Confluent Hypergeometric Functions in connection to some special functions: Incomplete Gamma Function, Error Function, Laguerre Polynomial Function, Hermite Polynomial Function, Bessel Function, and Whittaker Function via series expansion within the broader framework based on literature.

Results and Discussion

The Confluent hypergeometric function, a type of hypergeometric function, is one of the well-known forms of special functions. In this paper, we attempt to show the nexus of Confluent hypergeometric function to some other special functions: Incomplete Gamma function, Error function, Laguerre polynomial functions, Hermite polynomial functions, Whittaker functions through the series expansion.

i) Nexus of Confluent hypergeometric function to the Incomplete gamma function.

Let $q = p+1$ & $x = -x$ in [1.3]

Then, $_1F_1(p; p+1; -x) = \sum_{n=0}^{\infty} \frac{(p)_n}{(p+1)_n} \times \frac{(-x)^n}{n!}$

$$_1F_1(p; p+1; -x) = \sum_{n=0}^{\infty} \frac{p(p+1)(p+2).....(p+n-1)}{(p+1)(p+2)(p+3)........(p+n)} \times \frac{(-x)^n}{n!}$$

$$_1F_1(p; p+1; -x) = \sum_{n=0}^{\infty} \frac{p}{p+n} \times \frac{(-x)^n}{n!} \qquad [1.4]$$

According to Gupta (1999) lower incomplete gamma function in the series expansion form is

$$\Gamma(p,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \frac{(x)^{p+n}}{(p+n)}$$

This can be expressed in the integral form as

$$\Gamma(p,x) = \int_0^x t^{p-1} \sum_{k=0}^{\infty} \frac{(-t)k}{k!} dt$$

Or, $$\Gamma(p,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \int_0^x t^{p+k-1} dt$$

Or, $$\Gamma(p,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{x^{p+k}}{p+k}$$

Or, $$\Gamma(p,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^k}{p+k} \times x^p \right)$$

Or, $$px^{-p}\Gamma(p,x) = px^{-p}x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{x^k}{p+k}$$

Or, $$px^{-p}\Gamma(p,x) = \sum_{k=0}^{\infty} \frac{p}{p+k} \times \frac{(-x)^k}{k!}$$

Or, $$px^{-p}\Gamma(p,x) = \sum_{n=0}^{\infty} \frac{p}{p+n} \times \frac{(-x)^n}{n!}$$

From [1.4] $px^{-p}\Gamma(p,x) =_1 F_1(p;p+1;-x)$

$$\therefore {}_1F_1(p;p+1;-x) = px^{-p}\Gamma(p,x)$$

Thus, the nexus between the Confluent hypergeometric function and Incomplete gamma function is established.

ii)   Nexus of Confluent hypergeometric function to Laguerre polynomial Function

Let $p = -p \ \& \ q = q+1$ in [1.3], we get

$${}_1F_1(-p;q+1;x) = \sum_{n=0}^{\infty} \frac{(-p)_n}{(q+1)_n} \times \frac{(x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{p!}{(p-n)!} \cdot \frac{q!}{(q+n)!} \right) \frac{(x)^n}{n!} \qquad [1.5]$$

According to Abramowitz & Stegun (1972) & Gupta (1999), the series expansion of the Laguerre polynomial function is

$$L_p^q(x) = \sum_{n=0}^{\infty}(-1)^n [{}^{p+q}_{p-k}] \times \frac{x^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \left(\frac{(p+q)!}{(p-n)!(q+n)!} \right) \frac{x^n}{n!} \qquad [1.6]$$

Dividing [1.5] by [1.6], we get

$$\frac{{}_1F_1\left[-p;q+1;x\right]}{L_p^q(x)} = \frac{p!q!}{(p+q)!}$$

Or,
$$\frac{{}_1F_1\left[-p;q+1;x\right]}{L_p^q(x)} = \frac{p!}{(q+1)_n}$$

$$\therefore {}_1F_1\left[-p;q+1;x\right] = \frac{p!}{(q+1)_n} L_p^q(x)$$

Again, when $p = -p \ \& \ q = 1$, the equation [1.3] reduces to a simple Laguerre polynomial as

$${}_1F_1\left[-p;1;x\right] = \sum_{n=0}^{\infty} \frac{(-p)_n}{(1)_n} \times \frac{x^n}{n!}$$

Then,
$${}_1F_1\left[-p;1;x\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \dfrac{p!}{(p-n)!}}{n!} \times \frac{x^n}{n!}$$

Or,
$${}_1F_1\left[-p;1;x\right] = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \cdot \frac{p!}{(p-n)!} \right) \frac{x^n}{n!}$$

Or,
$${}_1F_1\left[-p;1;x\right] = \frac{p!}{1!} L_p^0(x)$$

$$\therefore {}_1F_1\left[-p;1;x\right] = \frac{p!}{1!} L_p(x)$$

In the same manner, the other 27 results can be derived (Rainville, 1960). Thus, the nexus between the Confluent hypergeometric function and the Laguerre polynomial function is established.

iii) Nexus of Confluent hypergeometric function to the Error function

According to Abramowitz & Stegun (1972) & Gupta (1999), the Error function is

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} dt$$

$$= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^x t^{2k} dt$$

$$= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{x^{2k+1}}{2k+1} \qquad [1.7]$$

Let $p = \frac{1}{2}, q = \frac{3}{2}, x = -x^2$, the equation [1.3] reduces to

Or, $\quad {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} \times \frac{(-x^2)^k}{k!}$

Or, $\quad {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = \sum_{k=0}^{\infty} \frac{\left(\frac{(2k)!}{2^{2k}k!}\right)}{\left(\frac{(2k+1)!}{2^{2k}k!}\right)} \times \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \times \frac{(-x^2)^k}{k!}$

Or, $\quad \frac{2x}{\sqrt{\pi}} {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = \frac{2x}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \times \frac{x^{2k}}{k!}$

Or, $\quad \frac{2x}{\sqrt{\pi}} {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \times \frac{x^{2k+1}}{k!}$

Using [1.7]

$$\frac{2x}{\sqrt{\pi}} {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = erf(x)$$

$$\therefore \quad {}_1F_1(\frac{1}{2};\frac{3}{2};-x^2) = \frac{\sqrt{\pi}}{2x} erf(x)$$

Thus, the nexus between the Confluent hypergeometric function and the Error function is established.

iv) Nexus of Confluent hypergeometric function to Hermit polynomial function

According to Bell (1968) & Gupta (1999), the series expansion of the Hermit polynomial function is

$$H_n(x) = n! \sum_{k=0}^{n/2} \frac{(-1)^k}{k!(n-2k)!} \times (2x)^{n-2k} \qquad [1.8]$$

Let $p = -p$, the series terminates at k = n

Then, $${}_1F_1\left[-p; q; x\right] = \sum_{k=0}^{n} \frac{(-p)_k}{(q)_k} \times \frac{x^k}{k!} \qquad [1.9]$$

Replacing n by 2n

$$H_{2n}(x) = (2n)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n-2k)!} \times (2x)^{2n-2k} \qquad [1.10]$$

Let m = n − k, then k = n − m. When k = 0, then m = n and k = n, m = 0

$$H_{2n}(x) = (2n)! \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!(2m)!} \times (2x)^{2m}$$

$$H_{2n}(x) = (2n)!(-1)^n \sum_{m=0}^{n} \frac{(-1)^m 2^{2m}}{(n-m)!(2m)!} \times x^{2m} \qquad [1.11]$$

Also, a series for ${}_1F_1(-p; \frac{1}{2}, x^2) = \sum_{n=0}^{N} \frac{(-p)_m}{(\frac{1}{2})_m} \times \frac{x^{2m}}{m!}$

$$= \sum_{m=0}^{n} \frac{(-1)^m p!}{(p-m)!} \cdot \frac{2^{2m} m!}{(2m)!} \times \frac{x^{2m}}{m!}$$

$$= \sum_{m=0}^{n} \frac{(-1)^m p! 2^{2m}}{(p-m)!(2m)!} \times x^{2m}$$

Replacing *p* by *n*,

$$= \sum_{m=0}^{n} \frac{(-1)^m n! 2^{2m}}{(n-m)!(2m)!} \times x^{2m}$$

From [1.11]

$$H_{2n}(x) = (2n)!(-1)^n \frac{1}{n!} \left[{}_1F_1(-p; \frac{1}{2}; x^2) \right]$$

Again from [1.10]

$$H_{2n+1}(x) = (2n+1)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n+1-2k)!} \times (2x)^{2n+1-2k}$$

Let $m = n - k$, then $k = n - m$. When $k = 0, then\ m = n$ and when $k = n$, then $m = 0$

$$H_{2n+1}(x) = (2n+1)! \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!(2m+1)!} \times (2x)^{2m+1}$$

$$H_{2n+1}(x) = (2n+1)!(-1)^n \sum_{m=0}^{n} \frac{(-1)^m 2^{2m+1}}{(n-m)!(2m+1)!} \times x^{2m+1}$$

$$H_{2n+1}(x) = (2n+1)!(-1)^n 2x \sum_{m=0}^{n} \frac{(-1)^m 4^m}{(n-m)!(2m+1)!} \times x^{2m} \qquad [1.12]$$

Also, the series of

$${}_1F_1(-p;\frac{3}{2};x^2) = \sum_{m=0}^{n} \frac{(-p)_m}{(\frac{3}{2})_m} \times \frac{x^{2m}}{m!} = \sum_{m=0}^{n} \frac{(-1)^m n!}{(p-m)!} \cdot \frac{4^m m!}{(2m+1)!} \times \frac{x^{2m}}{m!}$$

$$= \sum_{m=0}^{n} \frac{(-1)^m n! 4^m}{(p-m)!(2m+1)!} \times x^{2m}$$

From [1.12], we get

$$H_{2n+1}(x) = (-1)^n (2p+1)! 2x [{}_1F_1(-p;\frac{3}{2};x^2)]$$

Hence, the nexus is established.

v)   Nexus of Confluent hypergeometric function to Bessel polynomial Function

According to Olver, F.W.J., and Maximon, L.C. (2010), the series expansion of the Bessel polynomial function is

$$J_p(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(p+l+1)} \times \left(\frac{x}{2}\right)^{p+2l} \qquad [1.13]$$

Now, consider the confluent hypergeometric function:

$$F(p+\frac{1}{2};2p+1;2ix) = \sum_{m=0}^{\infty} \frac{(p+\frac{1}{2})_m}{(2p+1)_m} \times \frac{(2ix)^m}{m!} \qquad [1.14]$$

Also, the exponential function is

$$e^{-ix} = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \qquad [1.15]$$

Multiplying [1.14] by [1.15]

$$e^{-ix} \times {}_1F_1(p+\frac{1}{2};2p+1;2ix) = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \times \sum_{m=0}^{\infty} \frac{(p+\frac{1}{2})_m}{(2p+1)_m} \times \frac{(2ix)^m}{m!}$$

Multiplying both sides by $\frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p$

$$\frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p \times e^{-ix} \times {}_1F_1(p+\frac{1}{2};2p+1;2ix) = \frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p \times \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \times \sum_{m=0}^{\infty} \frac{(p+\frac{1}{2})_m}{(2p+1)_m} \times \frac{(2ix)^m}{m!}$$

$$= \frac{1}{\Gamma(p+1)}\left(\frac{1}{2}\right)^p \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{(-i)^n(2i)^m(p+\frac{1}{2})_m}{n!m!(2p+1)_m}\right] \times x^{p+n+m}$$

Let $k = n+m$ then $n = k-m$

$$= \frac{1}{\Gamma(p+1)}\left(\frac{1}{2}\right)^p \times \sum_{k=0}^{\infty} x^{p+k} \times \sum_{m=0}^{k}\left[\frac{(-i)^{k-m}(2i)^m(p+\frac{1}{2})_m}{(k-m)!m!(2p+1)_m}\right]$$

Since $(i)^{k-m} \times (2i)^m = (-1)^{k-m} \times i^{k-m} \times 2^m \times i^m = (-1)^{k-m} \times 2^m \times i^k = (-1)^{k-m} \times 2^m \times (-1)^{k/2}$

and for the Bessel series, only even powers of x appear, so letting $k = 2l$

$$= \frac{1}{\Gamma(p+1)}\left(\frac{1}{2}\right)^p \times \sum_{l=0}^{\infty}(-1)^l x^{p+2l} \times \sum_{m=0}^{2l}\left[\frac{(-1)^m(2)^m(p+\frac{1}{2})_m}{(2l-m)!m!(2p+1)_m}\right] [1.16]$$

Comparing the coefficients of x^{p+2l} from [1.13] and [1.16]

$$\frac{(-1)^l}{l!\Gamma(p+l+1)} \times \frac{1}{2^{p+2l}} = \frac{1}{\Gamma(p+1)}\left(\frac{1}{2}\right)^p \times \sum_{l=0}^{\infty}(-1)^l \times \sum_{m=0}^{2l}\left[\frac{(-1)^m(2)^m(p+\frac{1}{2})_m}{(2l-m)!m!(2p+1)_m}\right] \quad [1.17]$$

After removing the common part, we get

$$\frac{\Gamma(p+1)}{l!\Gamma(p+l+1)} \times \frac{1}{2^{2l}} = \sum_{m=o}^{2l}\left[\frac{(-1)^m(2)^m(p+\frac{1}{2})_m}{(2l-m)!m!(2p+1)_m}\right]$$

Thus, from [1.16] and [1.17]

$$\frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p \times e^{-ix} \times {}_1F_1(p+\frac{1}{2};2p+1;2ix) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!\Gamma(p+l+1)} \times \left(\frac{x}{2}\right)^{p+2l}$$

$$\frac{1}{\Gamma(p+1)}\left(\frac{x}{2}\right)^p \times e^{-ix} \times {}_1F_1(p+\frac{1}{2};2p+1;2ix) = J_p(x)$$

Thus, the Nexus is established.

In the same manner, the nexus of Confluent hypergeometric function to Modified Bessel polynomial function can also be shown, and similarly, the nexus of Confluent hypergeometric function to Whittaker Function can be shown

$$M_{p,q}(x) = x^{q+1/2}e^{-x/2}{}_1F_1(q-p+1/2, 2q+1, x)$$

Thus, setting the suitable parameters in the Confluent hypergeometric function and applying appropriate transformations, different special functions are obtained under the single framework of series expansion.

Conclusion

By setting suitable parameters in the Confluent hypergeometric function

$$_1F_1\big[p;q;x\big] = \sum_{n=0}^{\infty}\frac{(p)_n}{(q)_n} \times \frac{x^n}{n!},$$

different special functions can be obtained, which shows the central role of the Confluent hypergeometric function in the nexus of the special functions. In this paper, we have explicitly shown the nexus of Confluent hypergeometric functions to other well-known special functions through the method of series expansion. Similarly, the nexus of Confluent hypergeometric functions to the other special functions can be shown via differential equations and integral

representations. This study helps comprehensive study of different special functions from different perspectives and helps to understand the theoretical concepts as well as their application across the various pure and applied mathematics, engineering, and other scientific disciplines.

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