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Research Article

Dynamic Analysis of a Closed Economy Using the IS-LM Model in Discrete Time

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Abstract

This study introduces an economic model with an exogenous tax rate to analyze changes in income and interest rates, employing the IS-LM (Investment-Saving and Liquidity Preference-Money Supply) framework in macroeconomics. The deterministic model, formulated as ODE system, is discretized into a discrete system using the Euler method. Equilibrium and stability analyses of the discrete system are performed analytically. Numerical simulations validate the analytical findings and visualize different dynamics. The results highlight the influence of parameter sensitivities on economic stability and activity.

Keywords: Stability; Discrete-time System; IS-LM Model; Interest Rate; Economic Activity

1 Introduction

The concept of equilibrium serves as a cornerstone in economic analysis, providing a critical benchmark for understanding a country's economic activities. In macroeconomics, equilibrium explains how the overall price level and total output of goods and services are determined within an economy. Achieving equilibrium is essential for sustainable economic growth, as it reflects an optimal allocation of resources without persistent market shortages or surpluses. This understanding equips policymakers with tools to design interventions that address imbalances and promote economic stability. Macroeconomic output, often quantified as gross domestic product (GDP), is typically estimated using three approaches: expenditure, value of final goods and services, and income. Among these, the expenditure method is most widely used due to its relative simplicity and reduced susceptibility to data noise. The IS-LM model further enriches macroeconomic analysis by illustrating the interplay between the goods market and the money market. This model emphasizes the relationship between real interest rates and GDP, demonstrating how equilibrium in these markets drives economic activity. Primarily applied to analyze short-term equilibrium, the IS-LM framework provides valuable insights into the effects of fiscal and monetary policies on output and interest rates. Various versions of IS-LM models are explored in textbooks such as (Fuente, 2000), (Shone, 2001), and (Asada *et al., 2010*) and scientific literature including (Szomolanyi *et al., 2016*), (Nozaki, 2016), and (Navarro *et al., 2022*).

In recent years, discrete models have gained prominence, driven by the discrete nature of data collection and the widespread adoption of numerical schemes for solving differential equations. These models are often derived from continuous systems using numerical methods such as the forward/backward Euler method, Mickens' non-standard discretization, or mixed implicit-explicit approaches. For example, Roger and Barnard employed the central difference method to discretize the continuous SIR epidemic model and analyze the local stability of equilibria (Roeger & Barnard, 2007). Similarly, Liu et al. developed four discrete mathematical models using the forward and backward Euler methods (Liu *et al., 2015*). In this study, we adopt the forward Euler method to discretize the ordinary differential equation system into a discrete framework. Significant research has also delved into the dynamic characteristics of economic models, particularly those incorporating specific investment functions. Studies such as (Krawiec & Szydlowski, 2001), (Zhang & Wei., 2004), and (Wu & Wang., 2019) have yielded valuable insights into the dynamical behavior of these models. For instance, Riad et al. proposed a delayed IS-LM model with a general investment function, extending business cycle models (Riad *et al., 2019*). Their work demonstrated how delays in investment implementation after decision-making could destabilize economic equilibrium, leading to fluctuations in macroeconomic dynamics. Similarly, Zhou and Lie introduced a generalized IS-LM model with time delays and two distinct lags in the capital accumulation equation, showing that these delays can trigger equilibrium instability and cyclical behavior (Zhou & Li., 2019). They highlighted the importance of accounting for time delays in economic systems. Furthermore, Hidayati et al. analyzed an IS-LM business cycle model incorporating time delays, finding that maximizing the total money supply stabilizes the system by reducing interest rates and capital stock growth (Hidayati *et al., 2019*).

In a flexible-price economy, equilibrium in the goods market is also influenced by interest rates, which affect investment and consumption. Lower interest rates typically stimulate consumption and investment, increasing aggregate demand, whereas higher rates have the opposite effect. This paper aims to establish a model for interpreting business cycles, focusing on the simultaneous determination of national income and interest rates in both the goods and banking sectors of a closed economy. It provides key insights into how monetary policy and interest rate adjustments interact with output levels and how various parameters influence income and interest rates. The remainder of this paper is structured as follows: Section 2 introduces the mathematical model. Section 3 presents the mathematical analysis, including equilibrium and stability analysis. Section 4 provides numerical simulations to validate the analytical findings. Finally, section 5 concludes the paper.

2 Mathematical model

The proposed model considers closed economy and is loosely the one proposed by (Gaspar, 2018). We begin by deriving the aggregate demand side, the "investment saving" and "liquidity preference money supply" (IS-LM) model. Starting with the goods market (IS curve) the consumption expenditure is given by

$$c = c_0 + b(1-\theta)y, \quad b > 0, \quad \theta < 1. \tag{1}$$

Where θ is the exogenous tax rate which is the tax rate determined by the government or an external agency, independent of the market forces (20% corporate tax rate or a flat VAT of 13%). Its purpose is to generate the revenue for the government spending like public goods, infrastructure and social services. It impacts disposable income, consumption saving and investment. The investment expenditure i is equal to

$$i = i_0 - \phi r, \quad \phi > 0. \tag{2}$$

where r is the rate of interest (cost of burrowing money or the return on saving). It is primarily determined by market forces and central bank policies thus fluctuate frequently based on the economic condition. It's purpose is to regulate the economic activity, control inflation and stabilize the currency. The income(GDP) that reflects the total goods and services produced and thus the total income earned in the economy is given by

$$y = c + i, \tag{3}$$

People set aside funds to be prepared for the emergencies or security from unforseen expenses or financial uncertainties (Precautionary motives) and also holds money for potential profit due to expected future change in the interest rates or asset price (Speculative motives). The money market (LM curve) capturing both precautionary and speculative motives is described as

$$m_d = k_1 y - k_2 r \tag{4}$$

This indicates that the demand for money increases as national income rises because of higher level of economic activity (more transactions) that require more money whereas the money demand decreases when interest rates are high as people prefer to hold less money and invest in interest- bearing assets. The real money supply which is the real purchasing power of the money is given by,

$$m_s = m, \text{ where m is constant,} \tag{5}$$

In order to study the disequilibrium dynamics, IS-LM model is recasted in the dynamic form guided by economic theory, (Raghavendra & Piiroinen, 2023). The quantity adjustment in the product market whenever investment exceeds saving, and price adjustment (an increase in the rate of interest) in the money market are formalized by the following system of first order differential equations as:

$$\begin{aligned} \frac{dy}{dt} &= c_1(i(y,r) - s(y,r)), \\ \frac{dr}{dt} &= c_2(m_d - m_s) \end{aligned} \tag{6}$$

where $c_1 > 0$ and $c_2 > 0$ are speed of the adjustment coefficients and $s(y,r) = y - c(y,r)$ is saving. Using equations (1), (2), (4), and (5) in equation (6) we get,

$$\begin{aligned} \frac{dy}{dt} &= c_1(i_0 + c_0 - \phi r - (b\theta - b + 1)y), \\ \frac{dr}{dt} &= c_2(k_1 y - k_2 r - m). \end{aligned} \tag{7}$$

The system (7) is discretized by using forward Euler method by substituting $\dot{y} = \frac{y_{t+1} - y_t}{h}$ and $\dot{r} = \frac{r_{t+1} - r_t}{h}$. The corresponding discrete system is given by

$$\begin{aligned} y_{t+1} &= y_t + hc_1(i_0 + c_0 - \phi r_t - (b\theta + 1 - b)y_t) \\ r_{t+1} &= r_t + hc_2(k_1 y_t - k_2 r_t - m). \end{aligned} \tag{8}$$

where h is the size of the time space that determines how closely the difference equation resembles with the ordinary differential equation. For small h this system approaches ordinary differential equation. The model describes how income/output and interest rate changes from one period to the next. We will analyze the system (8).

3 Mathematical analysis

3.1 Positivity and boundedness of Solution

Theorem 1. *The region in which every solution of model system (8) that begins in the positive quadrant confined is given by:*

$$\Omega = \left\{(y,r) \in \mathbb{R}^2_+ : \ 0 \leq y(t) \leq K, \ 0 \leq r(t) \leq (k_1 K - m)\right\},$$

where, $K = \dfrac{i_0 + c_0}{b\theta - b + 1}$

Table 1: Descriptions of parameters and their values involved in the system (8).

Parameters	Descriptions	values
i_0	Autonomous investment	1
c_0	Autonomous consumption	1
b	Marginal propensity to consumption (MPC)	0.7
ϕ	Sensitivity of investment expenditure to the interest rate	0.5
θ	Exogenous tax rate (a flat income tax)	0.5
c_1	Adjustment coefficient in the goods market	1
c_2	Adjustment coefficient in the money market	1
k_1	Sensitivity to money demand to income/output	0.5
k_2	Sensitivity for money demand to interest rate	0.5
m	Nominal money supply	0.5

3.2 Feasibility of equilibria

The intersection of the IS and LM curves determines the equilibrium point. At the equilibrium point the goods and money market both are in balance. That is saving equals investment and money demands equals money supply. The equilibrium of the system (8) is obtained by putting $y_{t+1} = y_t = y^*$ and $r_{t+1} = r_t = r^*$. From the 1^{st} equation of the system (8) we get,

$$r_t = \frac{i_0 + c_0 - (b\theta + 1 - b)y_t}{\phi}, \tag{9}$$

and from the 2^{nd} equation of the system (8) we get,

$$r_t = \frac{k_1 y_t - m}{k_2}, \tag{10}$$

From equations (9) and (10) we can obtain,

$$y_t = \frac{i_0 + c_0 + \dfrac{m\phi}{k_2}}{b\theta + 1 - b + \dfrac{k_1\phi}{k_2}} = y^*, \tag{11}$$

and

$$r_t = \frac{k_1 y^* - m}{k_2} = r^*. \tag{12}$$

3.3 Stability analysis

The Jacobian matrix of the system (8) is

$$J = \begin{bmatrix} 1 - hc_1(b\theta + 1 - b) & -hc_1\phi \\ hk_1c_2 & 1 - hk_2c_2 \end{bmatrix}$$

The eigenvalue λ is obtained from the following characteristic equation of the jacobian matrix

$$\lambda^2 - tr(J)\lambda + det(J) = 0 \tag{13}$$

where,

$$\begin{aligned} tr(J) &= 2 - b_1 \\ det(J) &= 1 - b_1 + b_2 \\ b_1 &= h[c_1(b\theta + 1 - b) + k_2 c_2] \\ b_2 &= h^2 c_1 c_2 [k_2(b\theta + 1 - b) + \phi k_1]. \end{aligned}$$

Theorem 2. *The interior equilibrium (y^*, r^*) exists if $b < 1$, is asymptotically stable if $b_2 < b_1 < 4$ and is globally asymptotically stable if $(c_1\phi - c_2k_1)^2 < 4c_1c_2k_2(b\theta + 1 - b)$.*

Proof. The system (8) is asymptotically stable if modulus of all eigenvalues of the Jacobian matrix $J(y^*, r^*)$ are less than one, and by *Jury condition* this is possible if and only if $|tr(J)| < 1 + det(J) < 2$, (Elaydi, 2005). From right inequality we have,

$$\begin{aligned} 1 + det(J) &< 2 \\ \Rightarrow b_2 < b_1. \end{aligned} \tag{14}$$

Also if $tr(J) > 0$ from the left inequality we have, $tr(J) < 1 + det(J)$. And this is true since $b_2 > 0$. Further if $tr(J) < 0$ we have from the left inequality

$$\begin{aligned} -tr(J) &< 1 + det(J) < 2 \\ \Rightarrow b_1 &< 4. \end{aligned} \tag{15}$$

From equation (14) and (15) we get the result.

Further, consider the Lyapunav function

$$V(y, r) = \frac{1}{2}(y - y^*)^2 + \frac{1}{2}(r - r^*)^2 \tag{16}$$

which is positive definite and vanishes at the equilibrium point.

$$\begin{aligned} \triangle V &= V(y_{t+1}, r_{t+1}) - V(y_t, r_t) \\ &= hc_1(y_t - y^*)(i_0 + c_0 - \phi r_t - (b\theta + 1 - b)y_t) + hc_2(r_t - r^*)(k_1y_t - k_2r_t - m) \\ &\quad + \frac{c_1^2h^2}{2}(y_t - y^*)(i_0 + c_0 - \phi r_t - (b\theta + 1 - b)y_t)^2 + \frac{c_2^2h^2}{2}(k_1y_t - k_2r_t - m)^2 \\ &= hc_1(y_t - y^*)(i_0 + c_0 - \phi r_t - (b\theta + 1 - b)y_t) + hc_2(r_t - r^*)(k_1y_t - k_2r_t - m) + O(h^2) \\ &\approx -hc_1(b\theta + 1 - b)(y_t - y^*)^2 - h(c_1\phi - c_2k_1)(y_t - y^*)(r_t - r^*) - hc_2k_2(r_t - r^*)^2 \\ &< 0 \end{aligned}$$

if $(c_1\phi - c_2k_1)^2 < 4c_1c_2k_2(b\theta + 1 - b)$ and smaller positive value of h. This means that the Lyapunav function decreases that confirms the global stability. □

Remark 1. *From the stability condition $b_2 < b_1$ we can easily obtain $\phi < \dfrac{k_2}{c_1k_1}$.*

4 Numerical simulations

In this section the analytical findings are justified through numerical simulations using MATLAB 2018. The description of the parameters and their values for the purpose of numerical simulation is given in the Table (1). The set of values in the Table (1) is used throughout the simulations unless otherwise stated. The eigenvalues of the Jacobian matrix for these parameter values are $-0.8250 \pm 0.6851i$. We get the stable interior equilibrium as $y^* = 2.1739$ and $r^* = 1.1739$. We have plotted the variation plots and phase portraits of state variables due to variation in the values of parameters of the system (8) taking $h = 0.01$. We get stable equilibrium in each cases as the stability conditions stated in the Theorem (2) are satisfied in each cases as shown in Table (2). Figures (1) and (2) show the variation plots with the change of exogenous

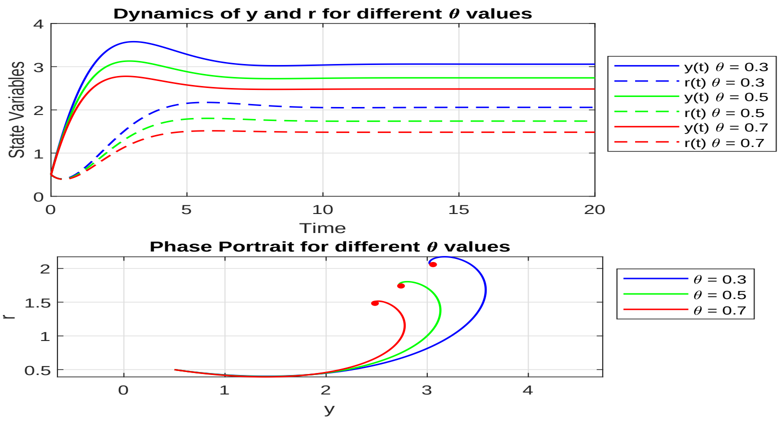

Figure 1: Variation plot and phase portrait for the system (8) for different value of θ.

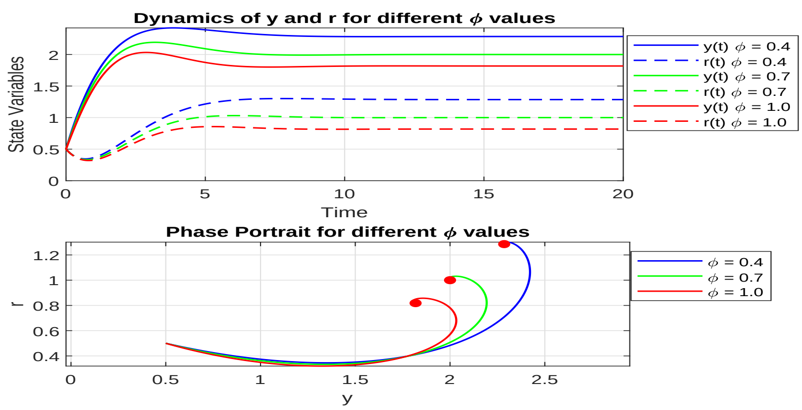

Figure 2: Variation plot and phase portrait for the system (8) for different value of ϕ.

tax rate, θ and sensitivity of investment expenditure to the interest rate, ϕ respectively. Both income and rate of interest decrease (increase) with the increase (decrease) in either exogenous tax rate or sensitivity of investment expenditure to the interest rate. The decrement in the rate of interest leads to increase in labor supply and investment that stimulate economic activity, potentially increasing income. Figure (3)

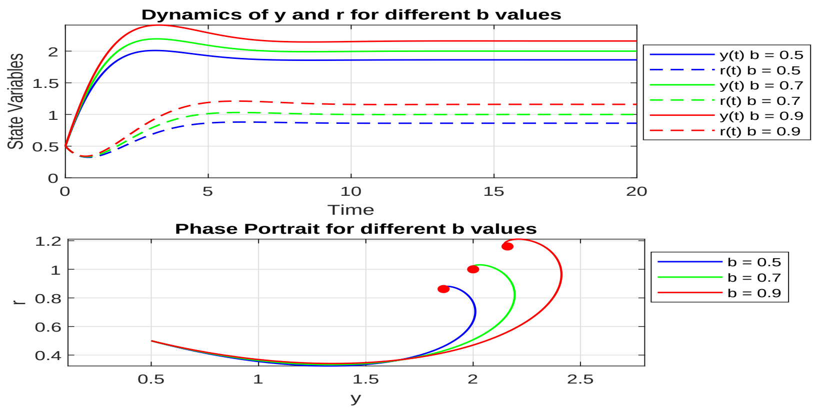

Figure 3: Variation plot and phase portrait for the system (8) for different value of b.

shows the variation in the income and interest rate for various values of marginal propensity to consume (the portion of additional income spend in consumption out of disposable income), b. The figure indicates that both income and rate of interest increase with the increase of marginal propensity to consume. Higher tax rates may discourage labor force participation, saving, or investment, potentially affecting productivity and economic output. Figure (4) shows the variation plot and phase portrait of the state variables with the change in the value of the sensitivity for money demand to income, k_1. It is found that income decreases with the increase in the value of k_1 whereas the rate of interest increases with the increase of k_1. The combined effect of lower income and a higher interest rate is a likely reduction in economic activity, with both consumption and investment adversely affected. Figure (5) reflects the global stability of the interior

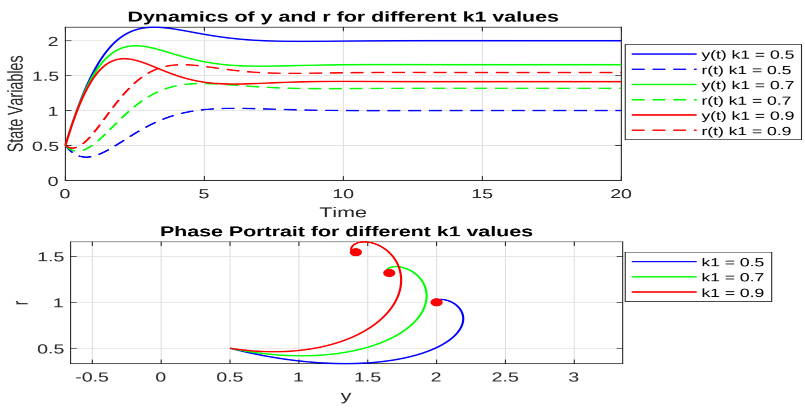

Figure 4: Variation plot and phase portrait for the system (8) for different value of k_1.

equilibrium (y^*, r^*). The global stability condition is satisfied for the chosen parameter values and so for different initial starts the trajectories converge to the equilibrium point $y^* = 1.6296$, and $r^* = 1.2296$.

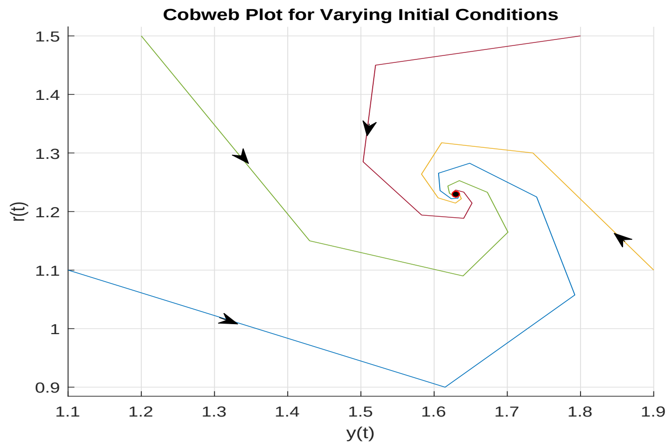

Figure 5: Global plot in the $y - r$ plane for the system (8), the value of parameters are same as in the Table (1) except m=0.2 and h=1.

Thus the system (8) is globally asymptotically stable. Figure (6) reveals the surface plot of the equilibrium income with the simultaneous change in the values of the sensitivity for money demand to income and to interest rate. It is found from the figure that the equilibrium level of income decreases due to increase of either k_1 or k_2. Thus simultaneous increase in the value of k_1 and k_2 causes low level of income. Figure (7) shows the surface plot of the equilibrium level of the rate of interest with respect to the change in the values of k_1 and k_2. From the figure it is clear that the value of r^* increases due to increase in the value of k_1 whereas decreases due to increase in the value of k_2. But if the values of the parameters k_1 and k_2 are simultaneously increased the equilibrium level r^* increases indicating that the parameter k_1 is more sensitive towards rate of interest than k_2. Figure (8) illustrates the variation in the equilibrium rate due to simultaneous change in the value of θ and ϕ. The surface plot indicates that the rate of interest decreases with simultaneous increase in the value of θ and ϕ.

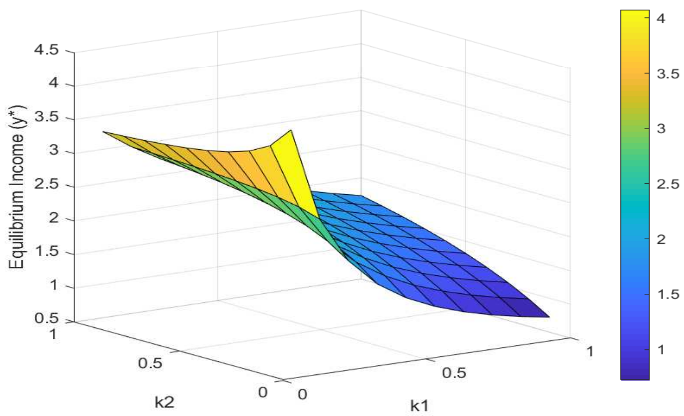

Figure 6: Surface plot for the equilibrium income with respect to k_1 and k_2 for the system (8),

Table 2: Equilibrium and Eigenvalues for different parameter values

Parameters	Equilibrium (y^*)	Equilibrium (r^*)	Eigenvalues	b_1	b_2
θ=0.3	3.0579	2.0579	-0.5050 ± 0.5916i	0.010100	0.000061
θ=0.5	2.7407	1.7407	-0.5750 ± 0.5868i	0.011500	0.000068
θ=0.7	2.4832	1.4832	-0.6450 ± 0.5736i	0.012900	0.000075
ϕ=0.4	2.2857	1.2857	-0.5750 ± 0.4409i	0.011500	0.000053
ϕ=0.7	2.0000	1.0000	-0.5750 ± 0.5868i	0.011500	0.000068
ϕ=1.0	1.8182	0.8182	-0.5750 ± 0.7031i	0.011500	0.000083
b=0.5	1.8621	0.8621	-0.6250 ± 0.4943i	0.012500	0.000073
b=0.7	2.0000	1.0000	-0.5750 ± 0.5868i	0.011500	0.000068
b=0.9	2.1600	1.1600	-0.5250± 0.5911i	0.010500	0.000063
k_1=0.5	2.0000	1.0000	-0.5750 ± 0.6851i	0.011500	0.000058
k_1=0.7	1.6564	1.3190	-0.5750 ± 0.6851i	0.011500	0.000068
k_1=0.9	1.4136	1.5445	-0.5750 ±0.6851i	0.011500	0.000078

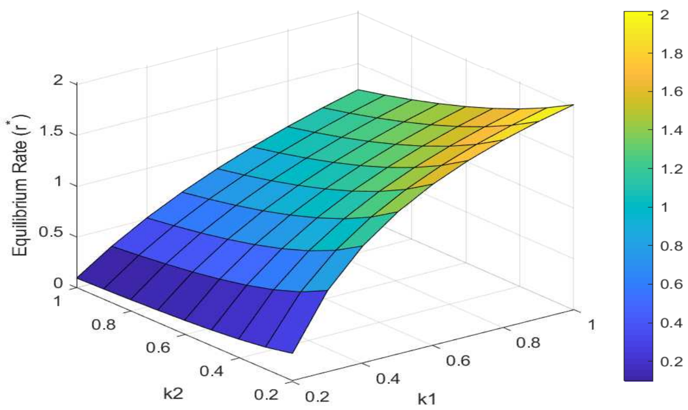

Figure 7: Surface plot for the equilibrium rate of interest with respect to k_1 and k_2 for the system (8),

5 Conclusion

The goods market represents the buying and selling of goods and services, while the money market captures the interplay between liquidity preference and the money supply. This study introduces a model to examine variations in income and interest rates. The ODE model is discretized using Euler's forward method. Existence of equilibrium and it's stability analysis is performed analytically and supported through numerical simulations. The impact of key parameters on income and rate of interest is examined, offering insights for fostering economic stability. The findings reveal that an increase in the exogenous tax rate θ and interest elasticity of investment ϕ reduces the disposable income and the rate of interest. The decline in income

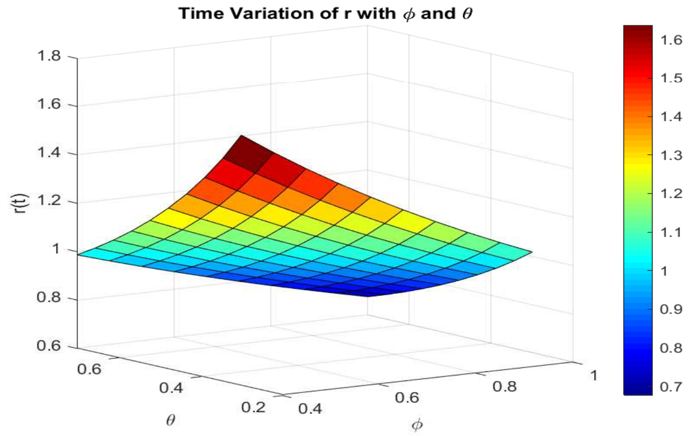

Figure 8: Surface plot for the equilibrium rate of interest with respect to θ and ϕ for the system (8),

suppresses the consumer's spending and households saving and reduction in the rate of interest also diminishes the saving which constraints funds available for the investment and adversely impacts the capital formation. The reduced consumption and investment lead to diminished aggregate demand, hampering economic growth, and in some cases, causing contraction. However, increase in the exogenous tax rate boosts the government revenue so during higher exogenous tax rate government should utilize the revenue appropriately into productive public investments to mitigate the short-term negative effects, potentially supporting long-term economic stability and growth. The analysis also highlights that a high marginal propensity to consume (MPC) amplifies both income and interest rate that may overheat the economy due to excessive consumption. In this situations encouraging savings through incentives such as tax breaks on savings accounts can minimize such volatility. Since households with lower income exhibit a higher MPC therefore addressing income inequality through progressive taxation, where tax rates rise with income, can reduce disparities and support economic balance. The study also reflects that economic stability is further achieved by keeping sensitivity of investment expenditure to the interest rate ϕ and sensitivity for money demand to income k_1 as low as possible where as sensitivity to money demand to interest rate k_2 higher for a given adjustment coefficient in the goods market c_1 whenever $0 < b < 1$. Thus, targeted policy interventions aligned with these insights can help foster sustained economic growth and stability.

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