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Surjective Monotone Mappings on Product Spaces

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Abstract

The role of duality mappings on the Surjectivity of monotone mapping is very crucial. In this paper we study some well-known results on the Surjectivity of monotone mappings on reflexive Banach spaces by Browder and its consequences. If $T: K \to 2^{K^*}$ *and* $S: L \to 2^{L^*}$ *are two maximal monotone mappings from reflexive Banach spaces* K *and* L*, we put the Surjectivity of the mapping* $H: K \times L \to 2^{K^* \times L^*}$ *defined by defined by*

$H(k,l) = \{(k^*, l^*): k^* \in Tk, l^* \in Sl\}, k \in K, l \in L.$

Keywords: *Duality mappings; Maximal monotone operators; Reflexive Spaces; Surjectivity*

Introduction

Suppose K is a real Banach space under the norm $||\cdot||$ and K^* is its dual norm. The duality pairing $\langle k^*, k\rangle$ gives the value of $k^* \in K^*$ at $k \in K$. An operator $T: K \to 2^{K^*}$ is multivalued whose effective domain is the set

$D(T) = \{k \in K: Tk \neq \phi\}$

and its graph is defined given by

$G(T) = \{(k, k^*) \in K \times K^*: k^* \in Tk\}.$

We call this multivalued mapping a monotone mapping if it satisfies the condition

$\langle x^* - y^*, x - y\rangle \geq 0$ for all $x^* \in Tx$, $y^* \in Ty, x, y \in D(T).$

If there is no further extension of such monotone mapping, they are termed as maximal monotone mappings. This situation can be stated mathematically as: for each $(x, x^*) \in X \times X^*$ such that

$\langle x^* - y^*, x - y\rangle \geq 0$ for all $(y, y^*) \in G(T)$, we have $(x, x^*) \in G(T)$.

The theory of maximal monotone operators on real reflexive Banach spaces has been studied quite extensively. For example, sufficient conditions on the operators on reflexive Banach spaces have been

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studied for the sum of two maximal monotone to be maximal monotone (see, for example, [10]). We recall that the sum of two monotone operators is always monotone but this property may fail to hold for maximal monotonicity in general. There are different contributions in this area by rockefellar [11], Attouch [1] etc.

Browder in [4] said that the reflexivity of the Banach space K is necessary for the Surjectivity condition of a maximal monotone mapping $T: K \to 2^{K^*}$. The Surjectivity fails in general Banach spaces which can be found in Gossez [10]. If K and L are two real Banach spaces and $T: K \to 2^{K^*}$ and $S: L \to 2^{L^*}$ are two maximal monotone operators then the operator $\mathrm{H}: \mathrm{K} \times \mathrm{L} \to 2^{\mathrm{K}^* \times \mathrm{L}^*}$ defined by

$$H(k,l) = \{(k^*, l^*): k^* \in Tk, l^* \in Sl\}, k \in K, l \in L$$

is maximal monotone (see [7]).

In this paper we consider both K and L as a real reflexive Banach spaces and we will discuss the Surjectivity of H.

Preliminaries

For any normed space K, a canonical mapping $c: K \to K^{**}$, defined by $c(k) = g_k \; \forall k \in K$. Then $g_k(f) = f(k)$ preserves the distance and hence it is an injective mapping. Clearly c is isomorphism from K into its range $R(c) \subseteq K^{**}$. A space K is said to be **reflexive** if $R(c) = K^{**}$.

The topology on the dual space K^*, weak topology is defined as following:

Definition 2.1 (Weak Topology, Rudin[12]). Suppose (L, τ) is a topological space. Let F denote a family of functions $g : K \to L$, where K is a nonempty set. On K, there is a weaker topology than the norm topology of K, called weak topology on K which makes all $g \in F$ continuous.

Definition 2.2 (Demi continuous, Cioranescu [8]) Let K and L are two real Banach spaces and K^*, L^* their duals. An operator $T: K \to L^*$ is demi continuous at $k \in D(T)$ if $k_n \in D(T), k_n \to k$ implies $T(k_n) \to Tk$.

Definition 2.3 (Duality Mapping). A mapping $J : K \to 2K^*$ defined by $J\varphi(k) = \{k^* \in K^* : \langle k^*, k\rangle = \| k^* \| \| k \|, \| k^* \| = \varphi(\| k \|)\}$ is called the duality mapping of K corresponding to the gauge function $\varphi : R_+ \to R_+$ such that φ(0) = 0, and $lim_{r\to\infty} \phi(r) = \infty$.

The duality mapping corresponding to the gauge function φ(r) = r for all r ∈ R+ is called the normalized duality mapping. The details of the following theorem can be found in Cioranescu [8].

To study the role of duality mappings on Surjectivity, we nee the concept of coercive mapping.

Definition 2.4 (Coercive mapping) A mapping $T: K \to 2^{K^*}$ is said to be coercive if there is a function $\phi: R_+ \to R_+$ with $\lim_{t\to\infty} \phi(t) = \infty$ and

$$\langle k^*, k\rangle \geq \phi(||k||) \cdot ||k||, \qquad \forall (k, k^*) \in G(T).$$

All the duality mappings are the examples of coercive mapping.

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The following theorem from Cioranescu [8], we have the following theorem which is essential to prove our result.

Theorem 2.5 (Cioranescu, [8]) Let K be a real Banach space and B be a closed convex set in K.

If $T: K \to 2^{K^*}$ is maximal monotone mapping with $D(T) \subseteq B$ and $S: B \to K^*$ is monotone, bounded, coercive and demi continuous operator, then $T + S$ is surjective.

Proof: let $k_0^* \in K^*$. Define $T'k = Tk - k_0^*$. Then the conditions mentioned above are fulfilled for T' and S. Then there exists $k_0 \in B$ such that

$\langle k_1^* + Sk_0, k_1 - k_0 \rangle \geq 0, \ \forall (k_1, k_1^*) G(T')$

i.e.

$\langle k^* - (k_0^* - Sk_0), k - k_0 \rangle \geq 0, \quad \forall (k, k^*) \in G(T).$

Since T is maximal monotone, we have that $(k_0, k_0^* - Sk_0) \in G(T)$ and this gives

$$k_0^* \in Tk_0 + Sk_0.$$

The following Surjectivity result for maximal monotone mappings is from Browder [5].

Theorem 2.6 Let K be a real reflexive Banach space and $T: K \to 2^{K^*}$ is a maximal monotone and coercive mapping then T is surjective.

The following theorem is due to Brezis [2] which was previously obtained by Browder [6] in the particular

Case when K^* is uniformly convex. This theorem was extended to general Banach space by Brezis-Browder [3].

Theorem 2.7 Let K be a reflexive Banach space and $T: K \to 2^{K^*}$ be a maximal monotone mapping, then T is surjective if and only if T^{-1} is locally bounded.

Results

Let X and Y represent two different Banach spaces with real values, and let X^* and Y^* represent their duals. Assume $F: X \to 2^{X^*}$ and $G: Y \to 2^{Y^*}$ are two duality mappings on X and Y with $D(F) = X$ and $D(G) = Y$ respectively. Then the mapping $H: (X \times Y) \to 2^{X^* \times Y^*}$ defined by

$$H(x, y) = \{(x^*, y^*): x^* \in F_x \ \& \ y^* \in G_x \ \forall x \in X \text{ and } y \in Y\} \text{ with}$$

domain as a whole space $X \times Y$ acquires some properties from that of F and G (see [4] for details).

Theorem 4.1 Let K and L be two real reflexive Banach spaces and $T: K \to 2^{K^*}$ and $S: L \to 2^{L^*}$ be two monotone mappings. Then a mapping $H: K \times L \to 2^{K^* \times L^*}$ is maximal monotone if and only if $H + J$ is surjective where J is normalized duality mapping.

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Proof: Being K and L reflexive, their cartesian product $K \times L$ is also reflexive. The normalized duality mappings are always monotone, bounded, demicontinuous and continuous. So, from Theorem 2.5 $H + J$ is surjective.

Conversely, let $H + J$ is surjective i.e., $R(H + J) = K^* \times L^*$ and H is not maximal monotone.
Then there exists $((k_1^*, l_1^*), (k_1, l_1) \in (K \times L) \times (K^* \times L^*)$ such that

$((k_1^*, l_1^*), (k_1, l_1) \in G(H)$

But

$$\langle (k_2^*, l_2^*) - (k_1^*, l_1^*), (k_2, l_2) - (k_1, l_1) \geq 0 \ \forall \left((k_2^*, l_2^*), (k_2, l_2)\right) \in G(H). \qquad (1)$$

From hypothesis there exists $(K_3^*, l_3^*), (k_3, l_3) \in G(H)$:

$$(k_1^*, l_1^*) + H(k_1, l_1) = (k_3^*, l_3^*) + H(k_3, l_3). \qquad (2)$$

Aking $(k_2, l_2) = (k_3, l_3)$ and $(k_2^*, l_2^*) = (k_3^*, l_3^*)$, we have from (1)

$$\langle (k_3^*, l_3^*) - (k_1^*, l_1^*), (k_3, l_3) - (k_1, l_1) \rangle \geq 0.$$

Hence

$$\langle H(k_3, l_3) - H(k_1, l_1), (k_3, l_3) - (k_1, l - 1) \rangle = 0.$$

Since J is strictly monotone, it gives

$$(k_3, l_3) = (k_1, l_1) \in D(H).$$

Thus from (2)

$$(k_1^*, l_1^*) = (k_3^*, l_3^*) \in H(k_3, l_3) \in G(H)$$

i.e.

$$((k_1^*, l_1^*), (k_1, l_1)) \in G(H)$$

a contradiction. So H is maximal monotone. This completes the proof.

We have following convexity result (proof can be found in Cioranescu [8]).

Theorem 4.2 Let $T: K \to 2^{K^*}$ a maximal monotone mapping on the real reflexive Banach space K then $\overline{R(T)}$ and $\overline{D(T)}$ *are convex.*

With the direct consequence of this theorem, we have the following corollary.

Corollary 4.3 Let K and L be two real reflexive Banach spaces and $T: K \to 2^{K^*}$ and $S: L \to 2^{L^*}$ be two monotone mappings. Then for a mapping $H: K \times L \to 2^{K^* \times L^*}$, $\overline{R(H)}$ and $\overline{D(H)}$ are convex.

Conclusion

The reflexivity of a real Banach space K is essential to be the monotone mapping surjective. If $T: K \to 2^{K^*}$ and $S: L \to 2^{L^*}$ are two maximal monotone mappings from reflexive Banach spaces K and L, then the mapping $H: K \times L \to 2^{K^* \times L^*}$ defined by defined by

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