# On Generalised $k$-Lucas Sequences

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**Abstract:** *The $k$-Lucas sequence is companion sequence of $k$-Fibonacci sequence defined with the $k$-Lucas numbers which are defined with the recurrence relation $\mathcal{L}_{k,n+1} = k\mathcal{L}_{k,n} + \mathcal{L}_{k,n-1}$, with the initial conditions $\mathcal{L}_{k,0} = 2$, $\mathcal{L}_{k,1} = 2$, for $n \geq 1$. In this paper, we introduce a new generalisation $\mathcal{M}_{k,n}$ of $k$-Lucas sequence. We present generating functions and Binet formulas for generalized $k$-Lucas sequence, and state some binomial and congruence sums containing these sequences.*

**Keywords:** Fibonacci Sequence, $k$-Fibonacci Sequence, $k$-Lucas Sequence.

## 1 Introduction

The well known integer sequence, Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers have many interesting properties and applications in many research areas such as architecture, nature, engineering and art. The Lucas sequence is companion sequence of Fibonacci sequence defined with the Lucas numbers which are defined with the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. Binet's formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_n = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root $r_1$ is known as the golden ratio. The Fibonacci and Lucas sequences are generalised by changing the initial conditions or changing the recurrence relation. One of the generalizations of the Fibonacci sequence is $k$-Fibonacci sequence first introduced by Falcon and Plaza [5]. The $k$-Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. Falcon [6] defined the $k$-Lucas sequence which is companion sequence of $k$-Fibonacci sequence defined with the $k$-Lucas numbers which are defined with the recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. Binet's formulas for the $k$-Fibonacci and $k$-Lucas numbers are

$$F_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots $r_1$ and $r_2$ satisfy the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

The reader can refer to [1, 2, 3, 4, 7, 8, 9, 10, 11, 12] for properties and applications of $k$-Fibonacci and $k$-Lucas numbers.

In the present paper, our main aim is to define generalised $k$-Lucas sequence and derive the relations connecting the generalised $k$-Lucas sequence and companion sequence. We have adapted the methods of Carlitz [2] and Zhizheng Zhang [3] to the generalised $k$-Lucas sequence $\mathcal{M}_{k,n}$ and derived some fundamental and congruence identities for these generalised $k$-Lucas sequences.

## 2 Generalized $k$-Lucas Sequence $\mathcal{M}_{k,n}$

In this section, we establish certain basic properties of the generalized $k$-Lucas sequence.

**Definition 1.** For $n \geq 1$, the generalized $k$-Lucas sequence $\mathcal{M}_{k,n}$ is defined by the recurrence relation $\mathcal{M}_{k,n+1} = k\mathcal{M}_{k,n} + \mathcal{M}_{k,n-1}$, with $\mathcal{M}_{k,0} = 2$ and $\mathcal{M}_{k,1} = k + \delta$.

**Definition 2.** For $n \geq 1$, the companion sequence $\mathcal{N}_{k,n}$ of $\mathcal{N}_{k,n}$ is defined by the relation $\mathcal{N}_{k,n} = \mathcal{N}_{k,n+1} + \mathcal{N}_{k,n-1}$.

The characteristic equation of the initial recurrence relation of $\mathcal{M}_{k,n}$is same as $k$-Lucas sequence. It is interesting to observe that if we add $\delta$ with $k$ in initial condition of sequence $\mathcal{M}_{k,n}$ then it is multiplied with $F_{k,n}$ and $F_{k,n} + L_{k,n}$ in the identities

$$\mathcal{M}_{k,n} = \delta F_{k,n} + L_{k,n},$$
$$\mathcal{N}_{k,n} = \delta\left(F_{k,n} + L_{k,n}\right),$$

respectively.

**Theorem 3.** ***(Binet Formulas).*** *For $n \geq 1$,*

$$\mathcal{M}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r_2}{r_2}^n}{r_1 - r_2} \tag{1}$$

*and*

$$\mathcal{N}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n, \tag{2}$$

*where, $\bar{r_1} = \delta + \sqrt{\delta}$ and $\bar{r_2} = \delta - \sqrt{\delta}$.*

Next, we state certain basic properties of the generalized $k$-Lucas sequence, these properties can proved using (1) and (2).

**Theorem 4.** ***(Catalan's Identity).*** *For $n, r \geq 1$, we have*

$$\mathcal{M}_{k,n-r}\mathcal{M}_{k,n+r} - {\mathcal{M}_{k,n}}^2 = (-1)^{n-r}\delta(1-\delta)F_{k,r}^2.$$

**Theorem 5.** ***(Cassini's Identity).*** *For $n \geq 1$, we have*

$$\mathcal{M}_{k,n-1}\mathcal{M}_{k,n+1} - {\mathcal{M}_{k,n}}^2 = (-1)^{n+1}\delta(1-\delta).$$

**Theorem 6.** ***(d'Ocagene's Identity).*** *Let $n$ be any non-negative integer and $r$ a natural number. If $n \geq r+1$, then*

$$\mathcal{M}_{k,r}\mathcal{M}_{k,n+1} - \mathcal{M}_{k,r+1}\mathcal{M}_{k,n} = (-1)^n\delta(1-\delta)F_{k,r-n}.$$

**Theorem 7.** ***(Convolution Theorem).*** *For $n, r \geq 1$, we have*

$$\begin{aligned}\mathcal{M}_{k,r}\mathcal{M}_{k,n+1} + \mathcal{M}_{k,r-1}\mathcal{M}_{k,n} = \mathcal{M}_{k,n+r} &+ (\delta^2 + \delta - \sqrt{\delta})F_{k,n+r} \\ &+ (2\delta + \sqrt{\delta})L_{k,n+r}.\end{aligned}$$

**Theorem 8.** ***(Asymptotic Behaviour)***. *For $n, r \geq 1$, we have*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{k,n}}{\mathcal{M}_{k,n-r}} = r_1^r.$$

**Theorem 9.** *The generating function for the generalized $k$-Fibonacci sequence $\mathcal{M}_{k,tn}$ is*

$$\sum_{n=0}^{\infty} \mathcal{M}_{k,tn} x^n = \frac{x\mathcal{M}_{k,t} - 2xL_{k,t} + 2}{1 - xL_{k,t} + x^2(-1)^t}.$$

**Theorem 10.** *For $n \geq 3$, we have*

$$r_1^{n-2} < \mathcal{M}_{k,n}.$$

**Theorem 11.** *For $n, k \geq 1$*

1. $\sum_{i=1}^{n} \mathcal{M}_{k,i} = \frac{\mathcal{M}_{k,n+1} + \mathcal{M}_{k,n} - (2 + k + \delta)}{k}$,
2. $\sum_{i=1}^{n} \mathcal{M}_{k,2i} = \frac{\mathcal{M}_{k,2n+1} - (k + \delta)}{k}$,
3. $\sum_{i=1}^{n} \mathcal{M}_{k,2i-1} = \frac{\mathcal{M}_{k,2n} - 2}{k}$,
4. $\sum_{i=1}^{n} \mathcal{M}_{k,i}^2 = \frac{k\mathcal{M}_{k,n+1}\mathcal{M}_{k,n} - k^2 - \delta(2k-1) - 2\delta^2}{k^2}$.

**Proposition 12.** *For $n \geq 0$, the following identities hold for $\mathcal{M}_{k,n}$ and $\mathcal{N}_{k,n}$:*

1. $\mathcal{M}_{k,n} = \delta F_{k,n} + L_{k,n}$,
2. $\mathcal{N}_{k,n} = \delta\left[F_{k,n} + L_{k,n}\right]$,
3. $\mathcal{M}_{k,n} + \mathcal{M}_{k,n+4} = (k^2 + 2)\mathcal{M}_{k,n+2}$,
4. $\mathcal{N}_{k,n} + \mathcal{N}_{k,n+4} = (k^2 + 2)\mathcal{N}_{k,n+2}$,
5. $\mathcal{N}_{k,n} + \mathcal{N}_{k,n+2} = \delta\mathcal{M}_{k,n+1}$,
6. $\mathcal{M}_{k,n-3} + \mathcal{M}_{k,n+3} = (k^2 + 1)\mathcal{N}_{k,n}$,
7. $\mathcal{N}_{k,n-3} + \mathcal{N}_{k,n+3} = \delta(k^2 + 1)\mathcal{M}_{k,n}$,
8. ${\mathcal{N}_{k,n}}^2 - \delta{\mathcal{M}_{k,n}}^2 = 4(-1)^{n+1}\delta(1 - \delta)$,
9. $\mathcal{M}_{k,2n}\mathcal{M}_{k,2n+1} = \mathcal{M}_{k,4n+1} + \delta(F_{k,4n+1} + L_{k,4n+1}) - k(\delta - 1)$.

Throughout this paper, the symbol $\binom{n}{i_1, i_2, ...., i_{(n-1)}}$ is defined by $\frac{n!}{i_1! i_2! .... i_{(n-1)}! s!}$, where

$$s = n - (i_1 + i_2 + .... + i_{(n-1)}).$$

In next section, we explore certain properties of the generalized $k$-Lucas sequence $\mathcal{M}_{k,n}$.

# 3 THE MAIN RESULTS

**Lemma 13.** *Let $u = r_1$ or $r_2$, then*

*(a)* $u^2 = ku + 1$,

*(b)* $u^n = uF_{k,n} + F_{k,n-1}$,

(c) $u^{2n} = u^n L_{k,n} - (-1)^n$,

(d) $u^{tn} = u^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}}$,

(e) $u^{sn} F_{k,rn} - u^{rn} F_{k,sn} = (-1)^{sn} F_{k,(r-s)n}$.

**Theorem 14.** *For $n, r, s, t \geq 1$, we have*

(a) $\mathcal{M}_{k,n+t} = F_{k,n}\mathcal{M}_{k,t+1} + F_{k,n-1}\mathcal{M}_{k,t}$,

(b) $\mathcal{M}_{k,2n+t} = L_{k,n}\mathcal{M}_{k,n+t} - (-1)^n \mathcal{M}_{k,t}$,

(c) $\mathcal{M}_{k,sn+t} = \dfrac{F_{k,sn}}{F_{k,n}}\mathcal{M}_{k,n+t} - (-1)^n \dfrac{F_{k,(s-1)n}}{F_{k,n}}\mathcal{M}_{k,t}$,

(d) $\mathcal{M}_{k,sn+t}F_{k,rn} - \mathcal{M}_{k,rn+t}F_{k,sn} = (-1)^{sn}\mathcal{M}_{k,t}F_{k,(r-s)n}$.

**Theorem 15.** *For $n, r, s, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,2n} = \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{D}_{k,i}$,
2. $\mathcal{D}_{k,2n+t} = \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{D}_{k,i+t}$,
3. $\mathcal{D}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} \mathcal{D}_{k,i+t}$,
4. $\mathcal{D}_{k,2rn+t} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} L_{k,r}^i \mathcal{D}_{k,ri+t}$,
5. $\mathcal{D}_{k,trn+l} = \dfrac{1}{F_{k,r}^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)(r+1)} F_{k,(t-1)r}^{n-i} F_{k,tr}^i \mathcal{D}_{k,ri+l}$,
6. $\sum_{i=0}^{n} \binom{n}{i} (-1)^i \mathcal{D}_{k,r(n-i)+i+t} F_{k,r}^i = \mathcal{D}_{k,t} F_{k,r-1}^n$,
7. $\sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} \mathcal{D}_{k,ri+t} F_{k,r-1}^{(n-i)} = \mathcal{D}_{k,n+t} F_{k,r}^n$,
8. $\sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} F_{k,sm}^{(n-i)} F_{k,rm}^{(i)} \mathcal{D}_{k,m[rn+i(s-r)]+t} = (-1)^{smn} \mathcal{D}_{k,t} F_{k,(r-s)m}^n$.

**Lemma 16.** *Let $u = r_1$ or $r_2$, then*

1. $k + (k^2+1)u = u^3$,
2. $1 + ku + u^6 = L_{k,2}u^4$,
3. $1 + ku + u^{10} = L_{k,4}u^6$,
4. $1 + ku + u^{18} = L_{k,8}u^{10}$,
5. $1 + ku + u^{34} = L_{k,16}u^{18}$,
6. $1 + ku + u^{66} = L_{k,32}u^{34}$,
7. $1 + ku + u^{130} = L_{k,64}u^{66}$,
8. $1 + ku + u^{258} = L_{k,128}u^{130}$,

9. $1+ku+u^{514}=L_{k,256}u^{258}$,

10. $1+ku+u^{1026}=L_{k,512}u^{514}$,

11. $1+ku+u^{2050}=L_{k,1024}u^{1026}$.

*In general, if $L_{k,n}$ is $n^{th}$ k-Lucas sequence and $u=r_1$ or $r_2$, then*

$$1+ku+u^{2(2^{n+1}+1)}=L_{k,2^{n+1}}u^{2(2^n+1)}.$$

**Theorem 17.** *For $t\geq 1$ and $\mathcal{D}_{k,n}=\mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,t+3}=(k^2+1)\mathcal{D}_{k,t+1}+k\mathcal{D}_{k,t}$,

2. $\mathcal{D}_{k,t+4}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+6}}{L_{k,2}}$,

3. $\mathcal{D}_{k,t+6}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+10}}{L_{k,4}}$,

4. $\mathcal{D}_{k,t+10}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+18}}{L_{k,8}}$,

5. $\mathcal{D}_{k,t+18}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+34}}{L_{k,16}}$,

6. $\mathcal{D}_{k,t+34}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+66}}{L_{k,32}}$,

7. $\mathcal{D}_{k,t+66}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+130}}{L_{k,64}}$,

8. $\mathcal{D}_{k,t+130}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+258}}{L_{k,128}}$,

9. $\mathcal{D}_{k,t+258}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+514}}{L_{k,256}}$,

10. $\mathcal{D}_{k,t+514}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+1026}}{L_{k,512}}$,

11. $\mathcal{D}_{k,t+1026}=\dfrac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+2050}}{L_{k,1024}}$.

*In general, for $t\geq 1$, we have*

$$\mathcal{D}_{k,t+2^{n+1}+2}=\frac{\mathcal{D}_{k,t}+k\mathcal{D}_{k,t+1}+\mathcal{D}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}}.$$

**Theorem 18.** *For $n,t\geq 1$ and $\mathcal{D}_{k,n}=\mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t}=\sum\limits_{i+j+s=n}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,2}}^{i}\mathcal{D}_{k,4i+6j+t}$,

2. $\mathcal{D}_{k,n+t}=\sum\limits_{i+j+s=n}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,4}}^{i}\mathcal{D}_{k,6i+10j+t}$,

3. $\mathcal{D}_{k,n+t}=\sum\limits_{i+j+s=n}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,8}}^{i}\mathcal{D}_{k,10i+18j+t}$,

4. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,16}}^{i} \mathcal{D}_{k,18i+34j+t}$,

5. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,32}}^{i} \mathcal{D}_{k,34i+66j+t}$,

6. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,64}}^{i} \mathcal{D}_{k,66i+130j+t}$,

7. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,128}}^{i} \mathcal{D}_{k,130i+258j+t}$,

8. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,256}}^{i} \mathcal{D}_{k,258i+514j+t}$,

9. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,512}}^{i} \mathcal{D}_{k,514i+1026j+t}$,

10. $\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,1024}}^{i} \mathcal{D}_{k,1026i+2050j+t}$.

*In general, for $r, n, t \geq 1$, we have*

$$\mathcal{D}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} \mathcal{D}_{k,2^{r+1}(i+2j)+2(i+j)+t}.$$

**Theorem 19.** *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,2}}^{i} \mathcal{D}_{k,4i+j+t}$,

2. $\mathcal{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,4}}^{i} \mathcal{D}_{k,6i+j+t}$.

3. $\mathcal{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,8}}^{i} \mathcal{D}_{k,10i+j+t}$,

4. $\mathcal{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,16}}^{i} \mathcal{D}_{k,18i+j+t}$,

5. $\mathcal{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,32}}^{i} \mathcal{D}_{k,34i+j+t}$,

6. $\mathcal{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,64}}^{i} \mathcal{D}_{k,66i+j+t}$,

7. $\mathcal{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,128}}^{i} \mathcal{D}_{k,130i+j+t}$,

8. $\mathcal{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,256}}^{i} \mathcal{D}_{k,258i+j+t}$,

9. $\mathcal{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,512}}^{i} \mathcal{D}_{k,514i+j+t}$,

10. $\mathcal{D}_{k,2050n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,1024}}^{i} \mathcal{D}_{k,1026i+j+t}$.

*In general, for $r, n, t \geq 1$, we have*

$$\mathcal{D}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j (-1)^{j+s} L_{k,2^{r+1}}{}^i \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

**Theorem 20.** *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,4n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,2}{}^{-n} \mathcal{D}_{k,6i+j+t}$,
2. $\mathcal{D}_{k,6n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,4}{}^{-n} \mathcal{D}_{k,10i+j+t}$,
3. $\mathcal{D}_{k,10n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,8}{}^{-n} \mathcal{D}_{k,18i+j+t}$,
4. $\mathcal{D}_{k,18n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,16}{}^{-n} \mathcal{D}_{k,34i+j+t}$,
5. $\mathcal{D}_{k,34n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,32}{}^{-n} \mathcal{D}_{k,66i+j+t}$,
6. $\mathcal{D}_{k,66n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,64}{}^{-n} \mathcal{D}_{k,130i+j+t}$,
7. $\mathcal{D}_{k,130n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,128}{}^{-n} \mathcal{D}_{k,258i+j+t}$,
8. $\mathcal{D}_{k,258n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,256}{}^{-n} \mathcal{D}_{k,514i+j+t}$,
9. $\mathcal{D}_{k,514n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,512}{}^{-n} \mathcal{D}_{k,1026i+j+t}$,
10. $\mathcal{D}_{k,1026n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,1024}{}^{-n} \mathcal{D}_{k,2050i+j+t}$.

*In general, for $r, n, t \geq 1$, we have*

$$\mathcal{D}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^j L_{k,2^{r+1}}{}^{-n} \mathcal{D}_{k,(2^{r+1}+2)i+j+t}.$$

**Lemma 21.** *Let $u = r_1$ or $r_2$, then for $l_n = \sum_{i=1}^{n} L_{k,2^i}$ and $n, t \geq 1$, we have*

1. $1 + u^4 = l_1 u^2$,
2. $1 + u^8 = \frac{l_2}{l_1} u^4 = l_2 u^2 - \frac{l_2}{l_1}$,
3. $1 + u^{16} = \frac{l_3}{l_2} u^8 = \frac{l_3}{l_1} u^4 - \frac{l_3}{l_2} = l_3 u^2 - \frac{l_3}{l_1} - \frac{l_3}{l_2}$,
4. $1 + u^{32} = \frac{l_4}{l_3} u^{16} = \frac{l_4}{l_2} u^8 - \frac{l_4}{l_3} = \frac{l_4}{l_1} u^4 - l_4 \left[\frac{1}{l_2} + \frac{1}{l_3}\right] = l_4 u^2 - l_4 \left[\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right]$,

*5.* $1+u^{64}=\dfrac{l_5}{l_4}u^{32}=\dfrac{l_5}{l_3}u^{16}-\dfrac{l_5}{l_4}=\dfrac{l_5}{l_2}u^{8}-l_5\left[\dfrac{1}{l_3}+\dfrac{1}{l_4}\right]=\dfrac{l_5}{l_1}u^{4}-l_5\left[\dfrac{1}{l_2}+\dfrac{1}{l_3}+\dfrac{1}{l_4}\right]$
$=l_5u^2-l_5\left[\dfrac{1}{l_1}+\dfrac{1}{l_2}+\dfrac{1}{l_3}+\dfrac{1}{l_4}\right].$

*In general, we have*

$$1+u^{2^n}=\begin{cases}\dfrac{l_{n-1}}{l_{n-2}}u^{2^{n-1}};\\ \dfrac{l_{n-1}}{l_{n-t-1}}u^{2^{n-t}}-l_{n-1}\sum\limits_{i=2}^{t}\dfrac{1}{l_{n-i}}, & \text{If } t=2,3,4,\dots,n-2\ ;\\ l_{n-1}u^2-l_{n-1}\sum\limits_{i=2}^{n-1}\dfrac{1}{l_{n-i}}.\end{cases}$$

**Theorem 22.** *For* $l_n=\sum\limits_{i=1}^{n}L_{k,2^i}$, $n,t\geq 1$ *and* $\mathcal{D}_{k,n}=\mathcal{M}_{k,n}$ *or* $\mathcal{N}_{k,n}$, *we have*

*1.* $\mathcal{D}_{k,t+4}=l_1\mathcal{D}_{k,t+2}-\mathcal{D}_{k,t}$,

*2.* $\mathcal{D}_{k,t+8}=\dfrac{l_2}{l_1}\mathcal{D}_{k,t+4}-\mathcal{D}_{k,t}=l_2\mathcal{D}_{k,t+2}-(1+\dfrac{l_2}{l_1})\mathcal{D}_{k,t}$,

*3.* $\mathcal{D}_{k,t+16}=\dfrac{l_3}{l_2}\mathcal{D}_{k,t+8}-\mathcal{D}_{k,t}$,
$=\dfrac{l_3}{l_1}\mathcal{D}_{k,t+4}-(1+\dfrac{l_3}{l_2})\mathcal{D}_{k,t}$,
$=l_3\mathcal{D}_{k,t+2}-(1+\dfrac{l_3}{l_1}+\dfrac{l_3}{l_2})\mathcal{D}_{k,t}$,

*4.* $\mathcal{D}_{k,t+32}=\dfrac{l_4}{l_3}\mathcal{D}_{k,t+16}-\mathcal{D}_{k,t}$,
$=\dfrac{l_4}{l_2}\mathcal{D}_{k,t+8}-(1+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,
$=\dfrac{l_4}{l_1}\mathcal{D}_{k,t+4}-(1+\dfrac{l_4}{l_2}+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,
$=l_4\mathcal{D}_{k,t+2}-(1+\dfrac{l_4}{l_1}+\dfrac{l_4}{l_2}+\dfrac{l_4}{l_3})\mathcal{D}_{k,t}$,

*5.* $\mathcal{D}_{k,t+64}=\dfrac{l_5}{l_4}\mathcal{D}_{k,t+32}-\mathcal{D}_{k,t}$,
$=\dfrac{l_5}{l_3}\mathcal{D}_{k,t+16}-(1+\dfrac{l_5}{l_4})\mathcal{D}_{k,t}$,
$=\dfrac{l_5}{l_2}\mathcal{D}_{k,t+8}-(1+\dfrac{l_5}{l_3}+\dfrac{l_5}{l_4})\mathcal{D}_{k,t}$,
$=\dfrac{l_5}{l_1}\mathcal{M}_{k,t+4}-(1+\dfrac{l_5}{l_2}+\dfrac{l_5}{l_3}+\dfrac{l_5}{l_4})\mathcal{D}_{k,t}$,
$=l_5\mathcal{D}_{k,t+2}-(1+\dfrac{l_5}{l_1}+\dfrac{l_5}{l_2}+\dfrac{l_5}{l_3}+\dfrac{l_5}{l_4})\mathcal{D}_{k,t}$.

*In general, we have*

$$\mathcal{D}_{k,t+2^n}=\begin{cases}\dfrac{l_{n-1}}{l_{n-2}}\mathcal{D}_{k,t+2^{n-1}}-\mathcal{D}_{k,t};\\ \dfrac{l_{n-1}}{l_{n-t-1}}\mathcal{D}_{k,t+2^{n-s}}-l_{n-1}\sum\limits_{i=2}^{s}(1+\dfrac{1}{l_{n-i}})\mathcal{D}_{k,t}, & \text{If } s=2,3,4,\dots,n-2\ ;\\ l_{n-1}\mathcal{D}_{k,t+2}-l_{n-1}\sum\limits_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1)\mathcal{D}_{k,t}.\end{cases}$$

**Theorem 23.** *For* $l_n = \sum_{i=1}^{n} L_{k,2^i}$, $n, t \geq 1$ *and* $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ *or* $\mathcal{N}_{k,n}$, *we have*

1. $\mathcal{D}_{k,4n+t} = \sum_{i+j=n} \binom{n}{i} l_1^i (-1)^j \mathcal{D}_{k,2i+t}$,

2. $\mathcal{D}_{k,8n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_2}{l_1})^i (-1)^j \mathcal{D}_{k,4i+t}$,
$= \sum_{i+j=n} \binom{n}{i} l_2^i (-1)^j (\frac{l_1 + l_2}{l_1}) \mathcal{D}_{k,2i+t}$.

3. $\mathcal{D}_{k,16n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_2})^i (-1)^j \mathcal{D}_{k,8i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_3}{l_1})^i (-1)^j (1 + \frac{l_3}{l_2}) \mathcal{D}_{k,4i+t}$,
$= \sum_{i+j=n} \binom{n}{i} {l_3}^i (-1)^j (1 + \frac{l_3}{l_1} + \frac{l_3}{l_2}) \mathcal{D}_{k,2i+t}$.

4. $\mathcal{D}_{k,32n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_3})^i (-1)^j \mathcal{D}_{k,16i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_2})^i (-1)^j (1 + \frac{l_4}{l_3}) \mathcal{D}_{k,8i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_4}{l_1})^i (-1)^j (1 + \frac{l_4}{l_2} + \frac{l_4}{l_3}) \mathcal{D}_{k,4i+t}$,
$= \sum_{i+j=n} \binom{n}{i} {l_4}^i (-1)^j (1 + \frac{l_4}{l_1} + \frac{l_4}{l_2} + \frac{l_4}{l_3}) \mathcal{D}_{k,2i+t}$,

5. $\mathcal{D}_{k,64n+t} = \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_4})^i (-1)^j \mathcal{D}_{k,32i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_3})^i (-1)^j (1 + \frac{l_5}{l_4}) \mathcal{D}_{k,16i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_2})^i (-1)^j (1 + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,8i+t}$,
$= \sum_{i+j=n} \binom{n}{i} (\frac{l_5}{l_1})^i (-1)^j (1 + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,4i+t}$,
$= \sum_{i+j=n} \binom{n}{i} {l_5}^i (-1)^j (1 + \frac{l_5}{l_1} + \frac{l_5}{l_2} + \frac{l_5}{l_3} + \frac{l_5}{l_4}) \mathcal{D}_{k,2i+t}$.

*In general, we have*

$$\mathcal{D}_{k,2^r n+t} = \begin{cases} \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-2}})^i (-1)^j \mathcal{D}_{k,2^{r-1}i+t}; \\ \sum_{i+j=n} \binom{n}{i} (\frac{l_{r-1}}{l_{r-s-1}})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{D}_{k,2^{n-s}i+t}, \\ \text{If } s = 2, 3, 4, \ldots, n-2 \,; \\ \sum_{i+j=n} \binom{n}{i} (l_{r-1})^i (-1)^j (\sum_{h=2}^{s} (1 + \frac{l_{r-1}}{l_{r-h}})^j \mathcal{D}_{k,2i+t}. \end{cases}$$

**Lemma 24.** *For* $t \geq 1$, *we have*

*(1)* $r_1^2 = r_1\sqrt{\delta} - 1$,
$r_2^2 = -r_2\sqrt{\delta} - 1$,

*(2)* $r_1^4 = (k^2 + 2)r_1\sqrt{\delta} - (k^2 + 3)$,
$r_2^4 = -(k^2 + 2)r_2\sqrt{\delta} - (k^2 + 3)$.

(3) $r_1^6 = (k^2+1)(k^2+3)r_1\sqrt{\delta} - (k^4+5k^2+5)$,
$r_2^6 = -(k^2+1)(k^2+3)r_2\sqrt{\delta} - (k^4+5k^2+5)$,

(4) $r_1^8 = (k^2+2)(k^4+4k^2+2)r_1\sqrt{\delta} - (k^6+7k^4+14k^2+7)$,
$r_2^8 = -(k^2+2)(k^4+4k^2+2)r_2\sqrt{\delta} - (k^6+7k^4+14k^2+7)$,

(5) $r_1^{10} = (k^4+3k^2+1)(k^4+5k^2+5)r_1\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$,
$r_2^{10} = -(k^4+3k^2+1)(k^4+5k^2+5)r_2\sqrt{\delta} - (k^2+3)(k^6+6k^4+9k^2+3)$.

*In general, we have*

$$r_1^{2t} = \frac{F_{k,2t}}{k}r_1\sqrt{\delta} - \frac{L_{k,2t-1}}{k},$$
$$r_2^{2t} = -\frac{F_{k,2t}}{k}r_2\sqrt{\delta} - \frac{L_{k,2t-1}}{k}.$$

**Lemma 25.** *For $t \geq 1$, we have*

(1) $r_1^3 = (k^2+3)r_1 - \sqrt{\delta}$,
$r_2^3 = (k^2+3)r_2 + \sqrt{\delta}$,

(2) $r_1^5 = (k^4+5k^2+5)r_1 - (k^2+2)\sqrt{\delta}$,
$r_2^5 = (k^4+5k^2+5)r_2 + (k^2+2)\sqrt{\delta}$,

(3) $r_1^7 = (k^6+7k^4+14k^2+7)r_1 - (k^2+1)(k^2+3)\sqrt{\delta}$,
$r_2^7 = (k^6+7k^4+14k^2+7)r_2 + (k^2+1)(k^2+3)\sqrt{\delta}$,

(4) $r_1^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_1 - (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,
$r_2^9 = (k^2+3)(k^6+6k^4+9k^2+3)r_2 + (k^2+2)(k^4+4k^2+2)\sqrt{\delta}$,

(5) $r_1^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_1 + (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$,
$r_2^{11} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)r_2 - (k^4+3k^2+1)(k^4+5k^2+5)\sqrt{\delta}$.

*In general, we have*

$$r_1^{2t+1} = \frac{L_{k,2t+1}}{k}r_1 - \frac{F_{k,2t}}{k}\sqrt{\delta},$$
$$r_2^{2t+1} = \frac{L_{k,2t+1}}{k}r_2 + \frac{F_{k,2t}}{k}\sqrt{\delta}.$$

**Theorem 26.** *For $s, t \geq 1$, we have*

1. $\mathcal{M}_{k,s+2} + \mathcal{M}_{k,s} = \mathcal{N}_{k,s+1}$,
   $\mathcal{N}_{k,s+2} + \mathcal{N}_{k,s} = \delta\mathcal{M}_{k,s+1}$,

2. $\mathcal{M}_{k,s+4} + (k^2+3)\mathcal{M}_{k,s} = (k^2+2)\mathcal{N}_{k,s+1}$,
   $\mathcal{N}_{k,s+4} + (k^2+3)\mathcal{N}_{k,s} = (k^2+2)\delta\mathcal{M}_{k,s+1}$,

3. $\mathcal{M}_{k,s+6} + (k^4+5k^2+5)\mathcal{M}_{k,s} = (k^2+1)(k^2+3)\mathcal{N}_{k,s+1}$,
   $\mathcal{N}_{k,s+6} + (k^4+5k^2+5)\mathcal{N}_{k,s} = (k^2+1)(k^2+3)\delta\mathcal{M}_{k,s+1}$,

4. $\mathcal{M}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{M}_{k,s} = (k^2+2)(k^4+4k^2+2)\mathcal{N}_{k,s+1}$,
   $\mathcal{N}_{k,s+8} + (k^6+7k^4+14k^2+7)\mathcal{N}_{k,s} = (k^2+2)(k^4+4k^2+2)\delta\mathcal{M}_{k,s+1}$,

5. $\mathcal{M}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{M}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\mathcal{N}_{k,s+1}$,
   $\mathcal{N}_{k,s+10} + (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{N}_{k,s} = (k^4+3k^2+1)(k^4+5k^2+5)\delta\mathcal{N}_{k,s+1}$.

*In general, we have*

$$\mathcal{M}_{k,s+2t} + \frac{L_{k,2t-1}}{k}\mathcal{M}_{k,s} = \frac{F_{k,2t}}{k}\mathcal{N}_{k,s+1}, \tag{3}$$

$$\mathcal{N}_{k,s+10} + \frac{L_{k,2t-1}}{k}\mathcal{N}_{k,s} = \frac{F_{k,2t}}{k}\delta\mathcal{N}_{k,s+1}. \tag{4}$$

*Remark* 27. Using $L_{k,2t-1} - F_{k,2t} = F_{k,2t-2}$ in (3), we get

$$\mathcal{M}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{M}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{M}_{k,s} = 0,$$
$$\mathcal{N}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{N}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{N}_{k,s} = 0.$$

**Theorem 28.** *For $s, t \geq 1$, we have*

1. $\mathcal{M}_{k,s+3} + \mathcal{N}_{k,s} = (k^2+3)\mathcal{M}_{k,s+1}$,
   $\mathcal{N}_{k,s+3} + \delta\mathcal{M}_{k,s} = (k^2+3)\mathcal{N}_{k,s+1}$,
2. $\mathcal{M}_{k,s+5} + (k^2+2)\mathcal{N}_{k,s} = (k^4+5k^2+5)\mathcal{M}_{k,s+1}$,
   $\mathcal{N}_{k,s+5} + \delta(k^2+2)\mathcal{M}_{k,s} = (k^4+5k^2+5)\mathcal{N}_{k,s+1}$,
3. $\mathcal{M}_{k,s+7} + (k^2+1)(k^2+3)\mathcal{N}_{k,s} = (k^6+7k^4+14k^2+7)\mathcal{M}_{k,s+1}$,
   $\mathcal{N}_{k,s+7} + \delta(k^2+1)(k^2+3)\mathcal{M}_{k,s} = (k^6+7k^4+14k^2+7)\mathcal{N}_{k,s+1}$,
4. $\mathcal{M}_{k,s+9} + (k^2+2)(k^4+4k^2+2)\mathcal{N}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{M}_{k,s+1}$,
   $\mathcal{N}_{k,s+9} + \delta(k^2+2)(k^4+4k^2+2)\mathcal{M}_{k,s} = (k^2+3)(k^6+6k^4+9k^2+3)\mathcal{N}_{k,s+1}$,
5. $\mathcal{M}_{k,s+11} + (k^4+3k^2+1)(k^4+5k^2+5)\mathcal{N}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{M}_{k,s+1}$,
   $\mathcal{N}_{k,s+11} + \delta(k^4+3k^2+1)(k^4+5k^2+5)\mathcal{M}_{k,s} = (k^{10}+11k^8+44k^6+77k^4+55k^2+11)\mathcal{N}_{k,s+1}$.

*In general, we have*

$$\mathcal{M}_{k,s+2t+1} + \frac{F_{k,2t}}{k}\mathcal{N}_{k,s} = \frac{L_{k,2t+1}}{k}\mathcal{M}_{k,s+1}, \tag{5}$$

$$\mathcal{N}_{k,s+2t+1} + \delta\frac{F_{k,2t}}{k}\mathcal{M}_{k,s} = \frac{L_{k,2t+1}}{k}\mathcal{N}_{k,s+1}. \tag{6}$$

*Remark* 29. Using $(k^2+3)F_{k,2t} - L_{k,2t-1} = F_{k,2t-2}$ in (5), we obtain

$$\mathcal{M}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{M}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\mathcal{N}_{k,s} = 0,$$
$$\mathcal{N}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{N}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\delta\mathcal{M}_{k,s} = 0.$$

**Theorem 30.** *For $n, s, t \geq 1$, we have*

1. $$\sum_{i=0}^{n}\binom{n}{i}\mathcal{M}_{k,2i+s} = \begin{cases} \delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ \delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
   $$\sum_{i=0}^{n}\binom{n}{i}\mathcal{N}_{k,2i+s} = \begin{cases} \delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ \delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd} \end{cases},$$
2. $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{M}_{k,4i+s} = \begin{cases} (k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
   $$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{N}_{k,4i+s} = \begin{cases} (k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd} \end{cases},$$

*3.* $\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{M}_{k,6i+s}$
$$= \begin{cases} (k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
$\sum_{i=0}^{n}\binom{n}{i}(k^4+5k^2+5)^{(n-i)}\mathcal{N}_{k,6i+s}$
$$= \begin{cases} (k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd} \end{cases},$$

*4.* $\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{M}_{k,8i+s}$
$$= \begin{cases} (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
$\sum_{i=0}^{n}\binom{n}{i}(k^6+7k^4+14k^2+7)^{(n-i)}\mathcal{N}_{k,8i+s}$
$$= \begin{cases} (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd} \end{cases},$$

*5.* $\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{M}_{k,10i+s}$
$$= \begin{cases} (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(k^6+6k^4+9k^2+3)^{(n-i)}\mathcal{N}_{k,10i+s}$
$$= \begin{cases} (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ (k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

*In general, for $n, s, t \geq 1$, we have*
$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(L_{k,2t-1})^{(n-i)}\mathcal{M}_{k,2ti+s} = \begin{cases} k^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;} \\ k^{-n}(F_{k,2t})^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(L_{k,2t-1})^{(n-i)}\mathcal{N}_{k,2ti+s} = \begin{cases} k^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;} \\ k^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 31.** *For $n, s, t \geq 1$, we have*

*1.* $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{M}_{k,2(n-i)+n} = \begin{cases} 2\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 2\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i\mathcal{N}_{k,2(n-i)+n} = \begin{cases} 2\delta^{\frac{n+2}{2}}, & \text{if } n \text{ is even;} \\ 2\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

*2.* $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{M}_{k,4(n-i)+n} = \begin{cases} 2(k^2+2)^n\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 2(k^2+2)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^4+5k^2+5)^i\mathcal{N}_{k,4(n-i)+n} = \begin{cases} 2(k^2+2)^n\delta^{\frac{n+2}{2}}, & \text{if } n \text{ is even;} \\ 2(k^2+2)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

*3.* $\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{M}_{k,6(n-i)+n}$
$$= \begin{cases} 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^6+7k^4+14k^2+7)^i\mathcal{N}_{k,6(n-i)+n}$$
$$=\begin{cases}2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^2+1)^n(k^2+3)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

*4.* $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{M}_{k,8(n-i)+n}$$
$$=\begin{cases}2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^2+3)^i(k^6+6k^4+9k^2+3)^i\mathcal{N}_{k,8(n-i)+n}$$
$$=\begin{cases}2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^2+2)^n(k^4+4k^2+2)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

*5.* $$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{M}_{k,10(n-i)+n}$$
$$=\begin{cases}2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}(k^{10}+11k^8+44k^6+44k^4+55k^2+11)^i\mathcal{N}_{k,10(n-i)+n}$$
$$=\begin{cases}2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k^4+3k^2+1)^n(k^4+5k^2+5)^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

*In general, for $n, s, t \geq 1$, we have*
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{M}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,}\end{cases}$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}k^{-i}(L_{k,2t+1})^i\mathcal{N}_{k,2t(n-i)+n}=\begin{cases}2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+2}{2}}, & \textit{if } n \textit{ is even;}\\ 2(k)^{-n}(F_{k,2t})^n\delta^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd.}\end{cases}$$

In next section, we prove some properties of the generalized $k$-Lucas sequence.

## The Proofs of the Main Results

**Proof of Lemma(13):** We prove only (a), (c) and (d) since the proofs of (b) and (e) are similar.

*Proof of (a):* Since $r_1$ and $r_2$ are roots of $r^2-kr-1=0$, then

$$r_1^2=kr_1+1, \tag{7}$$

$$r_2^2=kr_2+1. \tag{8}$$

This completes the proof of (a).

*Proof of (c):* From (b), we have

$$\begin{aligned}u^{2n}&=F_{k,n}u^{n+1}+u^nF_{k,n-1}\\&=F_{k,n}(uF_{k,n+1}+F_{k,n})+u^nF_{k,n-1}\\&=uF_{k,n}F_{k,n+1}+F_{k,n-1}u^n+F_{k,n}^2\\&=(u^n-F_{k,n-1})F_{k,n+1}+F_{k,n-1}u^n+F_{k,n}^2\\&=u^n(F_{k,n+1}+F_{k,n-1})+F_{k,n}^2-F_{k,n}F_{k,n-1}.\end{aligned}$$

Using $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain

$$u^{2n} = L_{k,n}u^n - (-1)^n.$$

This completes the proof of (c).

*Proof of (d):* If $u = r_1$, then we have

$$\begin{aligned} F_{k,tn}r_1^n - (-1)^n F_{k,(t-1)n} &= (\frac{r_1^{tn} - r_2^{tn}}{r_1 - r_2})r_1^n - (r_1r_2)^n(\frac{r_1^{(t-1)n} - r_2^{(t-1)n}}{r_1 - r_2}) \\ &= (\frac{r_1^n - r_2^n}{r_1 - r_2})r_1^{tn} \\ &= F_{k,n}r_1^{tn}. \end{aligned}$$

This completes the proof of (d).

The proofs of Theorems (17), (22) are similar. Hence, we prove only Theorem (14).

**Proof of Theorem(14):** We prove only (a), since the proofs of (b), (c) and (d) are similar.

*Proof of (1):* From 13(b), we have

$$r_1^n = F_{k,n}r_1 + F_{k,n-1}, \tag{9}$$

$$r_2^n = F_{k,n}r_2 + F_{k,n-1}. \tag{10}$$

Multiplying (9) by $\bar{r_1}r_1^t$, (10) by $\bar{r_2}r_2^t$ and subtracting, we obtain

$$\frac{\bar{r_1}r_1^{n+t} - \bar{r_2}r_2^{n+t}}{r_1 - r_2} = F_{k,n}(\frac{\bar{r_1}r_1^{t+1} - \bar{r_2}r_2^{t+1}}{r_1 - r_2}) + F_{k,n-1}(\frac{\bar{r_1}r_1^t - \bar{r_2}r_2^t}{r_1 - r_2}).$$

Hence, it gives that

$$\mathcal{M}_{k,n+t} = F_{k,n}\mathcal{M}_{k,t+1} + F_{k,n-1}\mathcal{M}_{k,t}.$$

This completes the proof of (a).

The proofs of Theorems (18)-(20) and (23) are similar. Hence, we prove only Theorem (15).

**Proof of Theorem(15):** We prove only (3), since the proofs of (1), (2) and (4)-(8) are similar.

*Proof of (3):*From 13(b), we have

$$r_1^r = F_{k,r}r_1 + F_{k,r-1}, \tag{11}$$

$$r_2^r = F_{k,r}r_2 + F_{k,r-1}. \tag{12}$$

Now, by the binomial theorem, we have

$$\bar{r_1}r_1^{rn} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i}\bar{r_1}r_1^i, \tag{13}$$

$$\bar{r_2}r_2^{rn} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i}\bar{r_2}r_2^i. \tag{14}$$

Now, by subtracting (13) from (14), we obtain

$$\frac{\bar{r_1}r_1^{rn+t} - \bar{r_2}r_2^{rn+t}}{r_1 - r_2} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} (\frac{\bar{r_1}r_1^{i+t} - \bar{r_2}r_2^{i+t}}{r_1 - r_2}).$$

Hence, it gives that

$$\mathcal{M}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} \mathcal{M}_{k,i+t}.$$

Now, by adding (13) and (14), we get

$$\bar{r_1}r_1^{rn+t} + \bar{r_2}r_2^{rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} (\bar{r_1}r_1^{i+t} + \bar{r_2}r_2^{i+t}).$$

Hence, it gives that

$$\mathcal{N}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^{i} F_{k,r-1}^{n-i} \mathcal{N}_{k,i+t}.$$

This completes the proof of (3).

**Proof of Lemma(16):** We prove only (1) and (2) since the proofs of (3)-(11) are similar.

*Proof of (1):* Using (7) and (8), we have

$$\begin{aligned} u^3 &= u^2 u \\ &= (ku+1)u \\ &= ku^2 + u \\ &= k(ku+1) + u \\ &= k^2 u + k + u \\ &= k + (k^2+1)u. \end{aligned}$$

This completes the proof of (1).

*Proof of (2):* Using (7) and (8), we have

$$\begin{aligned} 1 + ku + +u^6 &= u^2 + u^6 \\ &= u^2 + u^4(ku+1) \\ &= u^2 + ku^5 + u^4 \\ &= u^2 + ku^3(ku+1) + u^4 \\ &= u^2 + k^2u^4 + ku^3 + u^4 \\ &= (k^2+1)u^4 + ku^3 + u^2 \\ &= (k^2+1)u^4 + u^2(ku+1) \\ &= (k^2+1)u^4 + u^4 \\ &= (k^2+2)u^4 \\ &= F_{k,2}u^4. \end{aligned}$$

This completes the proof of (2).

The proofs of lemma (25) are similar. Hence, we prove only Lemma (24).

**Proof of Lemma(24):** We prove only (1) and (2) since the proofs of (3) - (5) are similar.

*Proof of (1):* Using (7) and (8), we have

$$\begin{aligned} r_1\sqrt{\delta}-1 &= r_1(r_1-r_2)-1 \\ &= r_1^2-r_1r_2-1 \\ &= r_1^2+1-1 \\ &= r_1^2. \end{aligned}$$

This completes the proof of (1).

*Proof of (2):* Using (7) and (8), we have

$$\begin{aligned} (k^2+2)r_1\sqrt{\delta}-(k^2+3) &= (k^2+2)r_1(r_1-r_2)-(k^2+3) \\ &= (k^2+2)(r_1^2-r_1r_2)-(k^2+3) \\ &= (k^2+2)(r_1^2+1)-(k^2+3) \\ &= r_1^2(k^2+2)+(k^2+2)-(k^2+3) \\ &= r_1^2k^2+2r_1^2-1 \\ &= r_1^2k^2+2(kr_1+1)-1 \\ &= r_1^2k^2+2kr_1+1 \\ &= r_1^2k^2+kr_1+kr_1+1 \\ &= (kr_1+1)(kr_1+1) \\ &= r_1^2r_1^2 \\ &= r_1^4. \end{aligned}$$

This completes the proof of (2).

The proofs of Theorems (26) and (28) are similar. Hence, we prove only Theorem (26).

**Proof of Theorem(26):** We prove only (2), since the proofs of (1) and (3)-(5) are similar.

*Proof of (2):*From 24(2), we have

$$r_1^4+(k^2+3)=(k^2+2)r_1\sqrt{\delta}, \tag{15}$$

$$r_2^4+(k^2+3)=-(k^2+2)r_2\sqrt{\delta}. \tag{16}$$

Multiplying (15) by $\bar{r_1}r_1^s$, (16) by $\bar{r_2}r_2^s$ and subtracting, we obtain

$$\frac{\bar{r_1}r_1^{s+4}-\bar{r_2}r_2^{s+4}}{r_1-r_2}+(k^2+3)\frac{\bar{r_1}r_1^s-\bar{r_2}r_2^s}{r_1-r_2} =(k^2+2)(\bar{r_1}r_1^{s+1}+\bar{r_2}r_2^{s+1})$$

Hence, it gives that

$$\mathcal{M}_{k,s+4}+(k^2+3)\mathcal{M}_{k,s} =(k^2+2)\mathcal{N}_{k,s+1}.$$

Multiplying (15) by $\bar{r_1}r_1^s$, (16) by $\bar{r_2}r_2^s$ and adding, we obtain

$$\bar{r_1}r_1^{s+4}+\bar{r_2}r_2^{s+4}+(k^2+3)(\bar{r_1}r_1^s+\bar{r_2}r_2^s)=(k^2+2)\delta(\frac{\bar{r_1}r_1^{s+1}-\bar{r_2}r_2^{s+1}}{r_1-r_2})$$

Hence, it gives that

$$\mathcal{N}_{k,s+4} + (k^2+3)\mathcal{N}_{k,s} = (k^2+2)\delta\mathcal{M}_{k,s+1}.$$

This completes the proof of (3).

The proofs of Theorems (30) and (31) are similar. Hence, we prove only Theorem (30).

**Proof of Theorem(30):** We prove only (2), since the proofs of (1)and (3)-(5) are similar.

*Proof of (2):*From 24(2), we have

$$r_1^4 + (k^2+3) = (k^2+2)r_1\sqrt{\delta},$$

$$r_2^4 + (k^2+3) = -(k^2+2)r_2\sqrt{\delta}.$$

Now, by the binomial theorem, we have

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_1}r_1^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_1}r_1^{n+s}), \tag{17}$$

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_2}r_2^{4i+s}) = (-1)^n(k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_2}r_2^{n+s}). \tag{18}$$

Now, by subtracting (17) from (18), we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\frac{\bar{r_1}r_1^{4i+s}-\bar{r_2}r_2^{4i+s}}{r_1-r_2}) = (k^2+2)^n\delta^{\frac{n}{2}}(\frac{\bar{r_1}r_1^{n+s}-(-1)^n\bar{r_2}r_2^{n+s}}{r_1-r_2}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{M}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^2+2)^n\delta^{\frac{n-1}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

Now, by adding (17) and (18), we get

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}(\bar{r_1}r_1^{4i+s}+\bar{r_2}r_2^{4i+s}) = (k^2+2)^n\delta^{\frac{n}{2}}(\bar{r_1}r_1^{n+s}+(-1)^n\bar{r_2}r_2^{n+s}).$$

Hence, it gives that

$$\sum_{i=0}^{n}\binom{n}{i}(k^2+3)^{(n-i)}\mathcal{N}_{k,4i+s} = \begin{cases}(k^2+2)^n\delta^{\frac{n}{2}}\mathcal{N}_{k,n+s}, & \text{if } n \text{ is even;}\\ (k^2+2)^n\delta^{\frac{n+1}{2}}\mathcal{M}_{k,n+s}, & \text{if } n \text{ is odd.}\end{cases}$$

This completes the proof of (3).

# 4 Some Congruence Properties of the Generalized $k$-Lucas Sequences

In this section, we established and proved certain congruence properties of the generalized $k$-Lucas sequence.

**Theorem 32.** *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,6j+t} \equiv 0 \pmod{L_{k,2}}$.
2. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,10j+t} \equiv 0 \pmod{L_{k,4}}$.
3. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,18j+t} \equiv 0 \pmod{L_{k,8}}$.
4. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,34j+t} \equiv 0 \pmod{L_{k,16}}$.
5. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,66j+t} \equiv 0 \pmod{L_{k,32}}$.
6. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,130j+t} \equiv 0 \pmod{L_{k,64}}$.
7. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,258j+t} \equiv 0 \pmod{L_{k,128}}$.
8. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,514j+t} \equiv 0 \pmod{L_{k,256}}$.
9. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,1026j+t} \equiv 0 \pmod{L_{k,512}}$.
10. $\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,2050j+t} \equiv 0 \pmod{L_{k,1024}}$.

*In general, for $r, n, t \geq 1$, we have*

$$\mathcal{D}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^n \mathcal{D}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

**Theorem 33.** *For $n, t \geq 1$ and $\mathcal{D}_{k,n} = \mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have*

1. $\mathcal{D}_{k,6n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,2}}$.
2. $\mathcal{D}_{k,10n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,4}}$.
3. $\mathcal{D}_{k,18n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,8}}$.
4. $\mathcal{D}_{k,34n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,16}}$.
5. $\mathcal{D}_{k,66n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,32}}$.
6. $\mathcal{D}_{k,130n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j}(-1)^n \mathcal{D}_{k,j+t} \equiv 0 \pmod{L_{k,64}}$.

*7.* $\mathcal{D}_{k,258n+t}-\sum\limits_{j=0}^{n}\binom{n}{j}k^j(-1)^n\mathcal{D}_{k,j+t}\equiv 0 \pmod{L_{k,128}}$.

*8.* $\mathcal{D}_{k,514n+t}-\sum\limits_{j=0}^{n}\binom{n}{j}k^j(-1)^n\mathcal{D}_{k,j+t}\equiv 0 \pmod{L_{k,256}}$.

*9.* $\mathcal{D}_{k,1026n+t}-\sum\limits_{j=0}^{n}\binom{n}{j}k^j(-1)^n\mathcal{D}_{k,j+t}\equiv 0 \pmod{L_{k,514}}$.

*10.* $\mathcal{D}_{k,2050n+t}-\sum\limits_{j=0}^{n}\binom{n}{j}k^j(-1)^n\mathcal{D}_{k,j+t}\equiv 0 \pmod{L_{k,1024}}$.

*In general, we have*

$$\mathcal{D}_{k,(2^{r+2}+2)n+t}-\sum_{j=0}^{n}\binom{n}{j}k^j(-1)^n\mathcal{D}_{k,j+t}\equiv 0 \pmod{L_{k,2^{r+1}}}.$$

The proofs of Theorems (32) and (33) are similar. Hence, we prove only Theorem (32).

**Proof of Theorem(32):** We prove only (1), since the proofs of (2)-(10) are similar.

*Proof of (1):* From Theorem (18;(1)), For $n,t\geq 1$ and $\mathcal{D}_{k,n}=\mathcal{M}_{k,n}$ or $\mathcal{N}_{k,n}$, we have

$$\begin{aligned}\mathcal{D}_{k,n+t} &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,2}}^i\mathcal{D}_{k,4i+6j+t}\\ &+\sum_{i+j+s=n;i=0}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,2}}^i\mathcal{D}_{k,4i+6j+t},\\ &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,2}}^i\mathcal{D}_{k,4i+6j+t}\\ &+\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathcal{D}_{k,6j+t}.\end{aligned}$$

$$\begin{aligned}&\mathcal{D}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathcal{D}_{k,6j+t}\\ &= \sum_{i+j+s=n;i\neq 0}\binom{n}{i,j}k^{-n}(-1)^{j+s}{L_{k,2}}^i\mathcal{D}_{k,4i+6j+t},\\ &\therefore L_{k,2}\quad \text{divides}\quad (\mathcal{D}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathcal{D}_{k,6j+t}),\\ &\therefore \mathcal{D}_{k,n+t}-\sum_{j=0}^{n}\binom{n}{j}k^{-n}(-1)^n\mathcal{D}_{k,6j+t}\equiv 0 \pmod{L_{k,2}}\end{aligned}$$

This completes the proof of (1).

## 5 Conclusion

In this paper, we present generating functions and Binet formulas for generalized $k$-Lucas sequence and its companion sequence. Also, derived some binomial and congruence sums containing these sequences.

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