Half-Cauchy Generalized Exponential Distribution: Theory and Application

Arun Kumar Chaudhary[1,*], Laxmi Prasad Sapkota[2], Vijay Kumar[3]

[1] Department of Management Science, Nepal Commerce Campus, Tribhuvan University, Nepal
[2] Department of Statistics, Tribhuvan Multiple Campus, Tribhuvan University, Palpa, Nepal
[3] Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, India

*Correspondence to: Arun Kumar Chaudhary, Email: akchaudhary1@yahoo.com

Abstract: *Half-Cauchy generalized exponential (HCGE) distribution is a novel distribution that we have proposed on in this paper. The quantiles, the measures of skewness based on quartiles, and the measures of kurtosis based on octiles, survival function, the probability density function, hazard function, cumulative distribution function, cumulative hazard function, are just a few of the crucial statistical properties we have derived for the proposed distribution. To estimate the parameters of the half-Cauchy generalized exponential distribution, the maximum likelihood estimation method has been applied. For the evaluation of the new distribution's potential, we have considered a real dataset and compared the goodness-of-fit attained by proposed distribution with some competing distribution. The suggested model fits the data much better and is more adaptable than some other models.*

Keywords: Estimation, Generalized exponential distribution, Half-Cauchy distribution, Quantile function, Hazard function

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1 Introduction

There are many continuous probability distributions found in literature related to probability and applied statistics. The areas like environmental, actuarial, and medical sciences, economics, life sciences, demography, finance, and insurance where most of the classical distributions have been extensively used for modeling real datasets for many years. However, there is a definite need for customized forms of more adaptable models for simulating real datasets that can handle a significant amount of kurtosis and skewness in a variety of practical areas like banking, survival analysis, and insurance for detail see Galambos and Kotz [11].

By bending the curve at the origin such that only non-negative values can be shown, we have taken into consideration the half-Cauchy distribution that results from the Cauchy distribution in this study. As it can forecast more frequent long-distance spreading events, the heavy-tailed half-Cauchy distribution was utilized by Shaw [25] as a modeling substitute for spreading distances. In order to simulate ringing data on two species in Ireland and Britain, Paradis et al. [19] also employed the half-Cauchy distribution. Assume that X has a half-Cauchy distribution and is a non-negative random variable. Consequently, its cumulative distribution function (CDF) can be stated as

$$G\left(x;\theta\right)=\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right),\ x>0,\theta>0 \tag{1}$$

and the probability density function (PDF) corresponding to (1) is

$$g\left(x;\theta\right)=\frac{2}{\pi}\left(\frac{\theta}{\theta^2+x^2}\right),\ x>0,\theta>0 \tag{2}$$

Last some years many researchers have used the half-Cauchy distribution as a parent model. The beta-half-Cauchy distribution is a modification of the half-Cauchy distribution developed by Cordeiro and Lemonte

[8], using the Marshall-Olkin transformation, Jacob and Jayakumar [13] modified the half-Cauchy distribution and studied the autoregressive process of first order and to perform a Bayesian analysis using a universal scale parameter, Polson and Scott [20] employed the half-Cauchy distribution as a prior. The gamma half-Cauchy distribution has introduced by Alzaatreh et al. [1]. Cordeiro et al. [9] have developed the family of distribution using half-Cauchy distribution as generalized odd half-Cauchy family of distribution. Chaudhary and Kumar [7] have developed half- Cauchy modified exponential distribution. Similarly another new distribution has been defined by Chaudhary et al. [3] using half-Cauchy family called half-Cauchy extended exponential distribution having flexible hazard function. We are therefore interested in creating new distributions utilizing the half-Cauchy family of distributions. Using the generalized exponential (GE) model, which has been defined by (Gupta and Kundu [12]), we have created a new distribution in this study. The CDF and PDF of the GE distribution are as follows

$$G(x) = \left(1 - e^{-\lambda x}\right)^{\alpha}; x > 0, \alpha > 0, \lambda > 0 \tag{3}$$

$$g(x) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}; x > 0, \alpha > 0, \lambda > 0 \tag{4}$$

Barreto-Souza et al. [2] have defined a new distribution using generalized exponential distribution called the beta generalized exponential distribution. Similarly, Maiti and Pramanik [16] have presented the odds generalized exponential-exponential distribution. Marshall-Olkin generalized exponential distribution was developed by Ristic and Kundu [23]. Another extension of generalized exponential distribution was presented by Dey et al. [10]. Chaudhary and Kumar [4] have presented logistic modified exponential distribution and further the extension of exponential distribution have been presented by Chaudhary and Kumar [5] called the logistic exponential extension distribution. New exponentiated exponential extension distribution was also presented by Chaudhary and Kumar [6].

The major objective of this work is to improve the GE distribution's flexibility by simply adding one additional parameter in order to get a satisfactory fit to real datasets. The article's remaining sections are organized as follows. Half-Cauchy generalized exponential distribution is defined in Section 2, and some of its statistical and mathematical features are covered and examined. Maximum likelihood estimators (MLE) method commonly used estimation technique that is employed in Section 3 to estimate the parameters of the proposed model. Section 4 presents the application of the suggested model. In Section 5, some concluding remarks are given.

2 The Half-Cauchy Generalized Exponential (HCGE) Distribution

We have used the half Cauchy family of distributions to extend the GE distribution in this study. The T-family defined by Ristic and Balakrishnan [22] is used to create the half-Cauchy family of distribution and its CDF may be written as

$$F\left(x\right) = 1 - \int\limits_{0}^{-\ln[G(x)]} r\left(t\right)\ dt. \tag{5}$$

In this case, $G(x)$ is the CDF of any baseline distribution and $r(t)$ is the PDF of any distribution. By using $r(t)$ as the PDF of the half-Cauchy distribution specified in (2) as below, we can define the half-Cauchy family of distribution

$$\begin{aligned} F\left(x\right) &= 1 - \int\limits_{0}^{-\ln[G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2 + t^2}\ dt \\ &= 1 - \frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\ln\left[G\left(x\right)\right]\right\} \end{aligned} \tag{6}$$

The PDF that corresponds to (6) can be written as

$$f(x) = \frac{2}{\pi}\frac{g(x)}{\theta G(x)}\left[1 + \left\{-\frac{1}{\theta}\log G(x)\right\}^2\right]^{-1} \tag{7}$$

The CDF and PDF of the HCGE distribution are now obtained by substituting (3) and (4) in (6) and (2), and they are, respectively, as follows

$$F(x) = 1 - \frac{2}{\pi} \arctan\left[-\frac{\alpha}{\theta} \ln\left(1 - e^{-\lambda x}\right)\right];\ x > 0,\ \alpha, \lambda, \theta > 0 \tag{8}$$

$$f(x) = \frac{2}{\pi} \frac{\alpha\lambda}{\theta} e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{-1} \left[1 + \left[-\frac{\alpha}{\theta} \ln\left(1 - e^{-\lambda x}\right)\right]^2\right]^{-1}; x > 0, \alpha, \lambda, \theta > 0 \tag{9}$$

It is clear from the Figure 1 that the PDF can have various types of shapes for different values of the parameters.

2.1 Reliability function

The HCGE distribution's reliability function is

$$R(x) = \frac{2}{\pi} \arctan\left[-\frac{\alpha}{\theta} \ln\left(1 - e^{-\lambda x}\right)\right];\ x > 0,\ \alpha, \lambda, \theta > 0 \tag{10}$$

2.2 Hazard rate function (HRF)

The HCGE's hazard rate function is

$$h(x) = \frac{\alpha\lambda}{\theta} e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{-1} \left\{\arctan\left[-\alpha/\theta \ln\left(1 - e^{-\lambda x}\right)\right]\right\}^{-1} \left[1 + \left[-\alpha/\theta \ln\left(1 - e^{-\lambda x}\right)\right]^2\right]^{-1} \tag{11}$$

We have plotted the graph of HRF of HCGE distribution in Figure 2 and found that it can have increasing or j-shaped or bathtub or up-side-down bathtub according to different values of the parameters.

2.3 Reversed hazard rate function (RHRF)

The reversed hazard rate function of HCGE distribution can be defined as

$$R_{HRF} = \frac{2}{\pi} \frac{\alpha\lambda}{\theta} e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{-1} \left\{1 - \frac{2}{\pi} \arctan\left[-\alpha/\theta \ln\left(1 - e^{-\lambda x}\right)\right]\right\}^{-1} \left[1 + \left[-\alpha/\theta \ln\left(1 - e^{-\lambda x}\right)\right]^2\right]^{-1} \tag{12}$$

Figure 1 shows the multiple PDF forms, we have plotted for varying HCGE distribution parameter values. We have also plotted the various shapes of HRF for different values of the parameters of HCGE distribution in Figure 2.

2.4 Cumulative hazard function (CHF)

The suggested model's CHF is defined as

$$\begin{aligned} H(x) &= \int_{-\infty}^{x} h(t)\, dt \\ &= -\log\left[1 - F(x)\right] \\ &= -\log\left[\frac{2}{\pi} \arctan\left\{-\frac{\alpha}{\theta} \ln\left(1 - e^{-\lambda x}\right)\right\}\right] \end{aligned} \tag{13}$$

2.5 Quantile function of the HCGE distribution

The quantile function can be obtained by inverting the CDF defined in (8) as

$$Q(u) = F^{-1}(u).$$

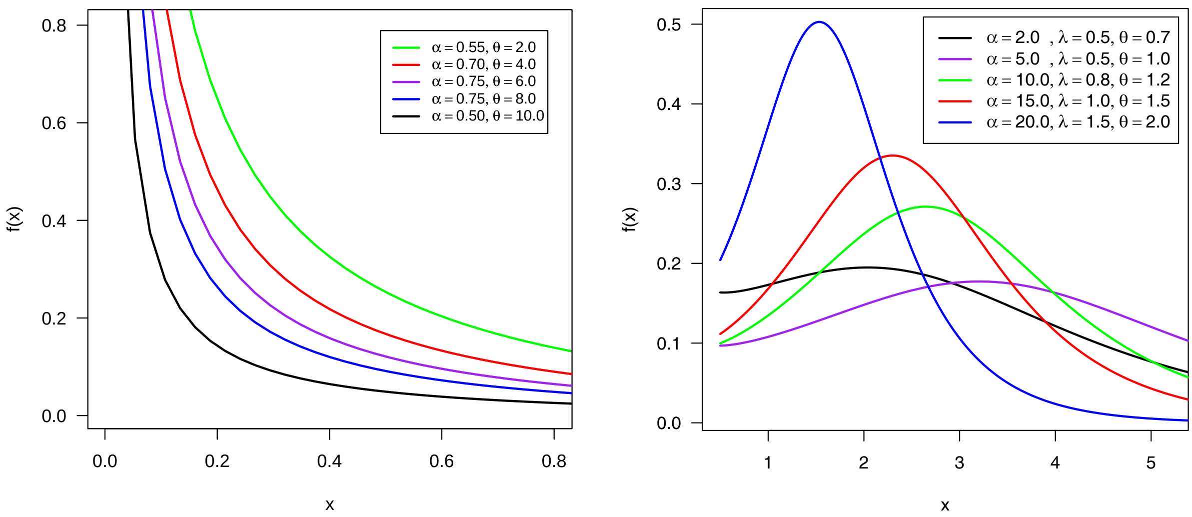

Figure 1: The graphs of PDF for different values of α, λ and θ.

Hence we can expressed the quantile function as

$$Q(u) = -\frac{1}{\lambda}\ln\left[1 - \exp\left\{-\frac{\theta}{\alpha}\tan\left(\frac{\pi\,(1-u)}{2}\right)\right\}\right];\; u \in (0,1) \tag{14}$$

Here u is the random variable that follows U (0,1). To generate the random numbers for HCGE distribution, we can utilize the following expression

$$x = -\frac{1}{\lambda}\ln\left[1 - \exp\left\{-\frac{\theta}{\alpha}\tan\left(\frac{\pi\,(1-v)}{2}\right)\right\}\right];\; \text{v} \in (0,1)\,.$$

2.6 Skewness and kurtosis for HCGE distribution

Skewness: The Bowley's coefficient of skewness based on quantiles can be achieved as

$$S\,(B) = \frac{Q\,(0.75) + Q\,(0.25) - 2Q\,(1/2)}{Q\,(3/4) - Q\,(0.25)}.$$

Kurtosis: Moors [17] used octiles to define the coefficient of kurtosis as

$$K_u\,(M) = \frac{Q\,(0.875) + Q\,(0.375) - Q\,(0.625) - Q\,(0.125)}{Q\,(3/4) - Q\,(1/4)}.$$

3 Parameter Estimation

3.1 Maximum likelihood estimation (MLE)

Here, we have discussed the ML estimators (MLE's) of the HCGE model are estimated by using MLE method. Let $\underline{x} = (x_1, \ldots, x_n)$ be a random sample drawn from a population of size n follows $HCGE(\alpha, \lambda, \theta)$. Then, the log likelihood function is

$$\ell(\alpha,\lambda,\theta|\underline{x}) = n\ln(2/\pi) + n\ln\left(\frac{\alpha\lambda}{\theta}\right) - \lambda\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\ln C(x_i) - \sum_{i=1}^{n}\ln\left[1 + \left[-\frac{\alpha}{\theta}\ln C(x_i)\right]^2\right] \tag{15}$$

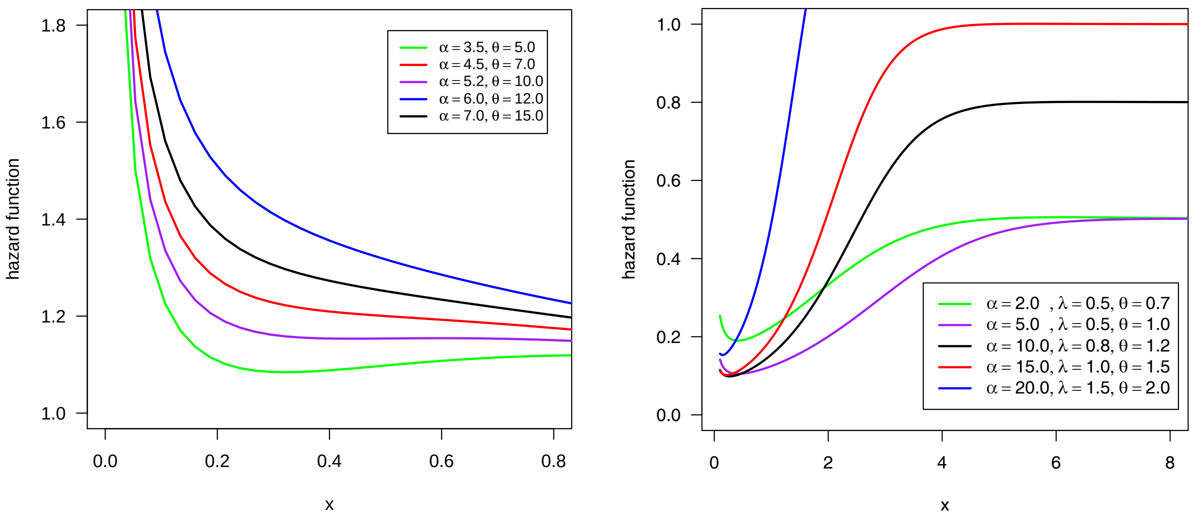

Figure 2: The graphs of HRF for different values of α, λ and θ.

Now, differentiating (15) with respect to α, λ and θ, we get

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \frac{2\alpha}{\theta^2} \sum_{i=1}^{n} \{\ln C(x_i)\}^2 \left[1 + \left[-\frac{\alpha}{\theta} \ln C(x_i)\right]^2\right]^{-1}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\lambda x_i} C(x_i)^{-1} - 2\left(\frac{\alpha}{\theta}\right)^2 \sum_{i=1}^{n} x_i e^{-\lambda x_i} C(x_i)^{-1} \ln C(x_i) \left[1 + \left[-\frac{\alpha}{\theta} \ln C(x_i)\right]^2\right]^{-1}$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{n}{\theta} + \frac{2\alpha^2}{\theta^3} \sum_{i=1}^{n} \left[C(x_i) \ln C(x_i) \left[1 + \left[-\frac{\alpha}{\theta} \ln C(x_i)\right]^2\right]^{-1}\right],$$

where $C(x_i) = 1 - e^{-\lambda x_i}$. The ML estimators for the $HCGE(\alpha, \lambda, \theta)$ model are obtained by solving for α, λ and θ, using the three non-linear equations $\frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \theta} = 0$. But normally, it is not possible to solve non-linear equations above, so with the aid of a suitable piece of software, one can solve them easily. If the parameter vector of $HCGE(\alpha, \lambda, \theta)$ be denoted by $\Theta = (\alpha, \lambda, \theta)$ and the associated MLE of Θ as $\hat{\Theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$ then, the resulting asymptotic normality is, $\left(\hat{\Theta} - \Theta\right) \to N_3 \left[0, (I(\Theta))^{-1}\right]$. The Fisher's information matrix, denoted by $I(\Theta)$ here, is provided by

$$I(\Theta) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}.$$

The asymptotic variance $(I(\Theta))^{-1}$ of the MLE has no practical significance because we do not know what Θ is. The estimated parameter values are therefore plugged into the approximate the asymptotic variance. The information matrix $I(\Theta)$ provided by the observed fisher information matrix $\hat{\Theta}$ is employed as a

$$O(\hat{\Theta}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \hat{\alpha}^2} & \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\theta}} \\ \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} & \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\theta}} & \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\theta}^2} \end{pmatrix}_{|(\hat{\alpha}, \hat{\lambda}, \hat{\theta})} = -H(\Theta)_{|(\Theta = \hat{\Theta})}.$$

Here, H denotes the Hessian matrix. The observed information matrix is built using the Newton-Raphson method to maximize likelihood, and the resulting variance-covariance matrix is as follows

$$\left[-H\left(\Theta\right)\big|_{(\Theta=\hat{\Theta})}\right]^{-1}=\begin{pmatrix}\operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{cov}(\hat{\alpha},\hat{\theta})\\ \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\theta})\\ \operatorname{cov}(\hat{\alpha},\hat{\theta}) & \operatorname{cov}(\hat{\lambda},\hat{\theta}) & \operatorname{var}(\hat{\theta})\end{pmatrix} \tag{16}$$

Let $Z_{c/2}$ denotes the upper percentile of the standard variate. So, using the asymptotic normality of MLEs, the following approximations of $100(1-c)\%$ confidence intervals for α, λ and θ can be created: $\hat{\alpha} \pm Z_{c/2}\sqrt{\operatorname{var}(\hat{\alpha})}$, $\hat{\lambda} \pm Z_{c/2}\sqrt{\operatorname{var}(\hat{\lambda})}$ and $\hat{\theta} \pm Z_{c/2}\sqrt{\operatorname{var}(\hat{\theta})}$.

4 Application to Real Dataset

In this part, we have used the real data set from a test that involved accelerated life for 59 conductors performed by Nelson and Doganaksoy [18]. Electro-migration, or the movement of atoms within the conductors of a circuit, is a cause of failures in microcircuits. No observations have been censored, and failure times are given in hours. The R software's **optim()** function is used to determine the MLEs of

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958,
4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024,
8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531,
7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923

the proposed distribution for detail see [15, 21]. The log likelihood value that we have determined is $l=-111.7792$. For the parameters α, λ and θ, we have shown the MLEs in Table 1 together with their standard errors (SE). The ML estimates are generated exclusively, as shown by the graphs of the profile log-

Table 1: SE and MLE for α, λ and θ of HCGE distribution

Parameter	**MLE**	**SE**
alpha	6.61405	5.48858
lambda	0.9352	0.11109
theta	0.01028	0.01178

likelihood function in Figure 3 for α, λ and θ. It is evident from the P-P and Q-Q plots in Figure 4 that the

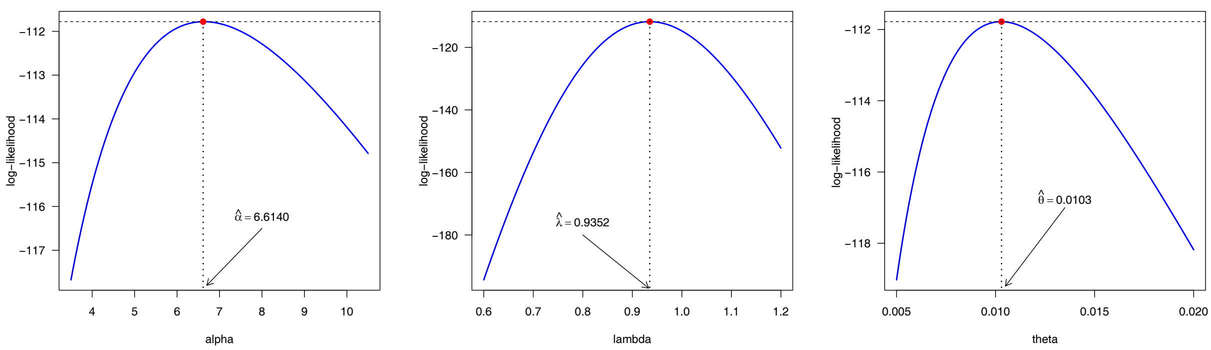

Figure 3: Graphs of Profile log-likelihood function for α, λ and θ.

suggested model closely matches the data. Using a real dataset that was previously utilized by researchers, we have demonstrated in this section the goodness-of-fit for the half-Cauchy generalized exponential model. We have chosen the five distributions such as Exponential power (EP) distribution by Smith and Bain [26],

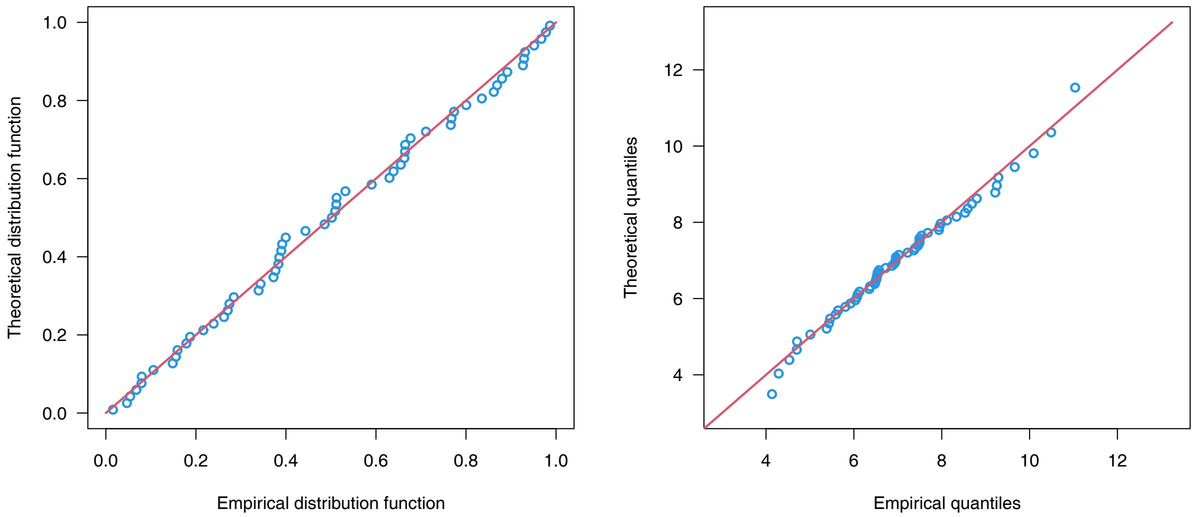

Figure 4: The P-P (left panel) and Q-Q (right panel) plots for the HCGE distribution.

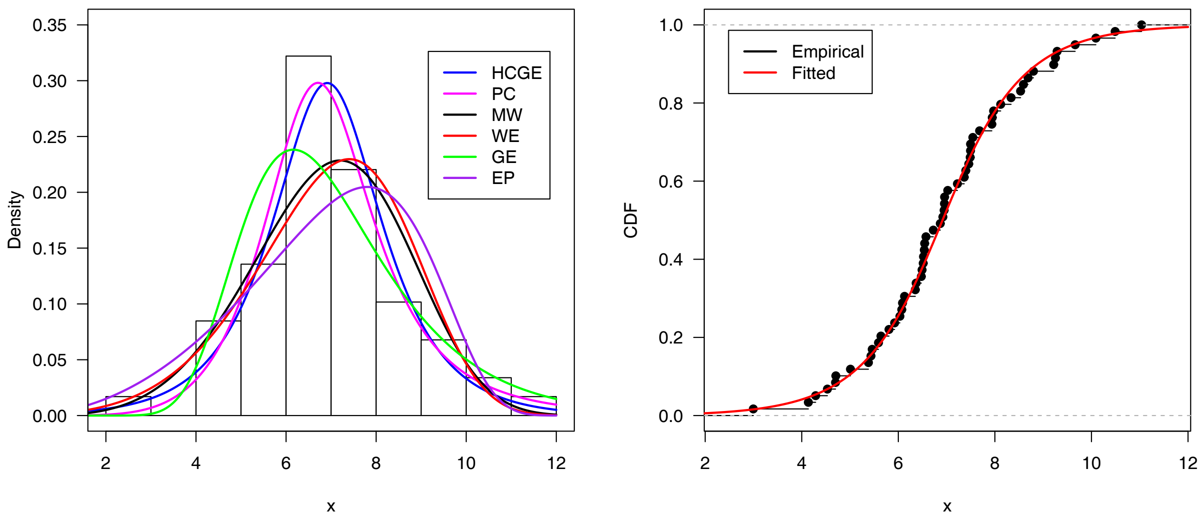

Figure 5: Empirical distribution function with estimated distribution function (right panel) and the Histogram and the density function of fitted distributions (left panel).

Generalized Exponential (GE) distribution by Gupta and Kundu [12], Modified Weibull (MW) by Lai et al. [14], Weibull Extension Model by Tang et al. [27] and Power Cauchy distribution by Rooks et al. [24] to contrast the recommended model's potential. We have calculated the corrected Akaike information criterion (CAIC), the Hannan-Quinn information criterion (HQIC) and the Bayesian information criterion (BIC) in order to assess the adequacy of the $HCGE(\alpha, \lambda, \theta)$ model. Table 2 displays these results. Figure 5 displays the histogram along with a few selected distributions, as well as the density functions of the fitted distributions, the empirical distribution function, and the estimated distribution function. Table 3 displays the results of the Cramer-Von Mises (CVM), Kolmogorov-Smirnov (KS), and Anderson-Darling

Table 2: AIC, CAIC, BIC, HQIC, and Log-likelihood (LL)

Model	AIC	BIC	CAIC	HQIC	LL
HCGE	229.558	235.791	229.995	231.991	-111.78
PC	228.183	232.338	228.397	229.805	-112.09
MW	231.044	237.276	231.48	233.477	-112.52
WE	233.349	239.582	233.786	235.782	-113.67
GE	233.895	238.05	234.109	235.517	-114.95
EP	237.003	241.158	237.21	238.625	-116.5

(AD) statistics in order to compare the goodness-of-fit of the suggested distribution to that of competing distributions. We draw the conclusion that the HCGE distribution has a much better fit to the data and more consistent and trustworthy findings when compared to others because it has a higher p-value and the lowest value of the test statistic.

Table 3: The goodness-of-fit statistics and the p-value

Model	***KS(p-value)***	***AD(p-value)***	***CVM(p-value)***
HCGE	0.0580(0.9821)	0.0247(0.9908)	0.1799(0.9950)
PC	0.0480(0.9982)	0.0199(0.9973)	0.1780(0.9953)
MW	0.0914(0.6738)	0.0821(0.6816)	0.4839(0.7626)
WE	0.1067(0.4796)	0.1154(0.5160)	0.6800(0.5751)
GE	0.1042(0.5103)	0.1173(0.5079)	0.7368(0.5282)
EP	0.1365(0.2021)	0.2398(0.2021)	1.3735(0.2098)

5 Conclusions

We have introduced a novel continuous distribution in this paper termed the half-Cauchy generalized exponential distribution. Along with a thorough consideration of some of the new distribution's mathematical and statistical characteristics, the exact equations for the skewness, and kurtosis, quantile function, survival function, hazard function, cumulative hazard function are addressed and examined. The most often used estimating approach, MLE method, is used to calculate the model parameters. Testing the appropriateness and application of the suggested distribution on a real data set revealed that it is far more flexible than some chosen distributions.

References

[1] Alzaatreh, A., Mansoor, M., Tahir, M. H., Zubair, M., and Ghazali, S. A., 2016, The gamma half-Cauchy distribution: Properties and applications, *Hacettepe Journal of Mathematics and Statistics*, 45(4), 1143-1159.

[2] Barreto-Souza, W., Santos, A. H., and Cordeiro, G. M., 2010, The beta generalized exponential distribution, *Journal of Statistical Computation and Simulation*, 80(2), 159-172.

[3] Chaudhary, A. K., Sapkota, L. P., and Kumar, V., 2022, Some properties and applications of half Cauchy extended exponential distribution, *Int J Stat Appl Math*, 7(4), 226-235.

[4] Chaudhary, A. K., and Kumar, V., 2020a, A study on properties and applications of logistic modified exponential distribution. *International Journal of Latest Trends in Engineering and Technology (IJLTET)*, 18(1), 19-29.

[5] Chaudhary, A. K., and Kumar, V., 2020b, The logistic exponential extension distribution with properties and applications, *International Journal of Latest Trends in Engineering and Technology (IJLTET)*, 18(2), 20-30.

[6] Chaudhary, A. K., and Kumar, V., 2020c, New exponentiated exponential extension distribution with properties and applications, *International Journal of Innovative Research in Science, Engineering and Technology (IJIRSET)*, 9(12), 11370-11379.

[7] Chaudhary, A. K., and Kumar, V., 2022, Half Cauchy-modified exponential distribution: properties and applications, *Nepal Journal of Mathematical Sciences*, 3(1), 47–58.

[8] Cordeiro, G. M., and Lemonte, A. J., 2011, The beta-half-Cauchy distribution, *Journal of Probability and Statistics*, 2011.

[9] Cordeiro, G. M., Alizadeh, M., Ramires, T. G., and Ortega, E. M., 2017, The generalized odd half-Cauchy family of distributions: properties and applications, *Communications in Statistics-Theory and Methods*, 46(11), 5685-5705.

[10] Dey, S., Alzaatreh, A., Zhang, C., and Kumar, D., 2017, A new extension of generalized exponential distribution with application to ozone data, *Ozone: Science & Engineering*, 39(4), 273-285.

[11] Galambos, J., and Kotz, S., 2006, *Characterizations of probability distributions: A unified approach with an emphasis on exponential and related models*, 675, Springer.

[12] Gupta, R. D., and Kundu, D., 1999, Generalized exponential distributions, *Australian and New Zealand Journal of Statistics*, 41(2), 173 - 188.

[13] Jacob, E., and Jayakumar, K., 2012, On half-Cauchy distribution and process, *International Journal of Statistika and Mathematika*, 3(2), 77-81.

[14] Lai, C., Xie, M., and Murthy, D., 2003, A modified weibull distribution, *IEEE Trans Reliab*, 52, 33-37.

[15] Mailund, T., 2017, *Functional programming in R: Advanced statistical programming for data science, Analysis and Finance*. Apress.

[16] Maiti, S. S., and Pramanik, S., 2015, Odds generalized exponential-exponential distribution, *Journal of data science*, 13(4), 733-753.

[17] Moors, J. J. A., 1988, A quantile alternative for kurtosis, *Journal of the Royal Statistical Society: Series D (The Statistician)*, 37(1), 25-32.

[18] Nelson, W., and Doganaksoy, N., 1995, Statistical analysis of life or strength data from specimens of various sizes using the power-(log) normal model, *Recent Advances in Life-Testing and Reliability*, 377-408.

[19] Paradis, E., Baillie, S. R, and Sutherland, W. J., 2002, Modeling large-scale dispersal distances, *Ecological Modelling*, 151(2-3), 279–292.

[20] Polson, N. G., and Scott, J. G., 2012, On the half-Cauchy prior for a global scale parameter, *Bayesian Analysis*, 7(4), 887-902.

[21] R Core Team, 2022, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

[22] Ristic, M. M., and Balakrishnan, N., 2012, The gamma-exponentiated exponential distribution, *Journal of Statistical Computation and Simulation*, 82(8), 1191-1206.

[23] Ristic, M. M., and Kundu, D., 2015, Marshall-Olkin generalized exponential distribution, *Metron*, 73(3), 317-333.

[24] Rooks, B., Schumacher, A., and Cooray, K., 2010, The power Cauchy distribution: derivation, description, and composite models, *NSF-REU Program Reports*.

[25] Shaw, M. W., 1995, Simulation of population expansion and spatial pattern when individual dispersal distributions do not decline exponentially with distance, *Proceedings of the Royal Society B: Biological Sciences*, 259, 243–248.

[26] Smith, R. M., and Bain, L. J., 1975, An exponential power life-test distribution, *Communications in Statistics*, 4, 469-481.

[27] Tang, Y., Xie, M., and Goh, T. N., 2003, Statistical analysis of a Weibull extension model, *Communications in Statistics-Theory and Methods*, 32(5), 913-928.