On Topological Structure of Total Paranormed Double Sequence Space \((\ell^2((X,||.||),\bar{\gamma},\bar{w}),G)\)

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Abstract: The aim of this paper is to introduce and study a new class \(\ell^2((X,||.||),\bar{\gamma},\bar{w})\) of double sequences with their terms in a normed space \(X\) as a generalization of the familiar sequence space \(\ell\). Besides the investigation of the condition pertaining to the containment relations of the class \(\ell^2((X,||.||),\bar{\gamma},\bar{w})\) of same kind in terms of \(\bar{\gamma}\) and \(\bar{w}\), our primary interest is to explore some of the preliminary results that characterize the linear topological structures of \(\ell^2((X,||.||),\bar{\gamma},\bar{w})\) when topologized it with suitable natural paranorm.

Keywords: Paranormed space, Sequence space, Double sequence

1 Introduction

We begin with recalling some notations and basic definitions that are used in this paper. The concept of paranorm is closely related to the linear metric space; see Wilansky [26] and its studies on sequence spaces were initiated by Maddox [9], and many others. Basarir and Altundag [1], Ghimire, & Pahari [2, 3], Parasar and Choudhary [15], Paudel and Pahari [18, 20, 21] and many others further studied the various types of paranormed sequence spaces.

A paranormed space \((X,G)\) is a linear space with the zero element \(\theta\) together with the function \(G : X \rightarrow \mathbb{R}^+\) (called a paranorm on \(X\)) which satisfy the following axioms

P.N1: \(G(\theta) = 0\);

P.N2: \(G(x) = G(-x)\) for all \(x \in X\);

P.N3: \(G(x+y) \leq G(x) + G(y)\) for all \(x, y \in X\); and

P.N4: Scalar multiplication is continuous

Note that the continuity of scalar multiplication is equivalent to

i. \(G(x_k) \to 0\) and \(\lambda_k \to \lambda\) as \(k \to \infty\) then \(G(\lambda_k x_k) \to 0\) as \(k \to \infty\), and

ii. \(\lambda_k \to 0\) as \(k \to \infty\) and \(x\) be any element in \(X\), then \(g(\lambda_k x) \to 0\) see Wilansky [26].

A paranorm is called total if \(G(x) = \theta\) implies \(x = \theta\).

Let \(X\) be a normed space over the field of complex numbers. Let \(\omega(X)\) denotes the space of all sequences \(\bar{x} = (x_i)\) with \(x_i \in X, i \geq 1\). We shall denote \(\omega(C)\) by \(\omega\). Any subspace \(S\) of \(\omega\) is then called a sequence space. A normed space valued sequence space or a generalized sequence space is a linear space of sequences with their terms in a normed space. Several workers like Gupta and Patterson [5], Kantam and Gupta [6], Kolk [7], Köthe [8], Maddox [10], and Pahari [12] etc. have introduced and studied some properties of vector and scalar-valued single sequence spaces, when sequences are taken from a Banach space.

The concept of various types of linear spaces of single sequences and their special kind of convergence was studied by several workers for instances we refer a few: Pahari [12, 13, 14], Pokharel, Pahari and Ghimire [22], Srivastava and Pahari [24, 25]. They also studied the various types of topological structures of vector valued sequence spaces defined by Orlicz function endowed with suitable natural paranorms and extended some of them in 2-normed spaces. Recently, Paudel, Pahari and et al. [17, 18, 19, 20, 21, 22] has extended the concepts of sequence space of complex numbers to the sequences of fuzzy real numbers.
On Topological Structure of Total Paranormed Double Sequence Space \((\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})), G\)

The theory of single sequence spaces has also been extended to the spaces of double sequences and studied by several workers. Gupta and Kanthan [4], Morics [11] and many others have made significant contributions and enriched the theories in this direction. In the recent years, Savas [22], Subramanian et al. [25] and many others have introduced and studied various types of double sequence spaces using Orlicz function.

2 The Class \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) of Double Sequences

Let \(\bar{\omega} = (w_{nk})\) and \(\bar{\omega} = (\bar{\omega}_{nk})\) be any double sequences of strictly increasing positive real numbers and \(\bar{\gamma} = (\gamma_{nk})\) and \(\bar{\mu} = (\mu_{nk})\), \(n, k \geq 1\) be double sequences of non-zero complex numbers. Let \((X,||.||), (Y,||.||)\) be Banach spaces over the field of complex numbers and \(B((X,||.||), Y)\) be the Banach space of all bounded linear operators from \((X,||.||)\) into \(Y\). The zero element of the Banach spaces \(X, Y, B((X,||.||), Y)\) will be denoted by \(\theta\).

Throughout the work, \(\sum \sum\) will denote

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \text{ in the sense that } \lim_{K \to \infty} \sum_{2 \leq n + k \leq K} \sum_{2 \leq n + k \leq K} \text{ as } n + k \to \infty
\]

We now introduce and study the following class of Banach space-valued double sequences

\[
\ell^2((X,||.||), \bar{\gamma}, \bar{\omega}) = \left\{ \bar{x} = (x_{nk}) : x_{nk} \in X, n, k \geq 1 \text{ and } \lim_{K \to \infty} \sum_{2 \leq n + k \leq K} \sum_{2 \leq n + k \leq K} ||\gamma_{nk} x_{nk}||^{w_{nk}} \to \theta \text{ as } n + k \to \infty \right\}
\]

Further, when \(\gamma_{nk} = 1\) for all \(n\) and \(k\), then \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) will be denoted by \(\ell^2((X,||.||), \bar{\omega})\) and when \(w_{nk}\) for all \(n\) and \(k\); then \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) will be denoted by \(\ell^2((X,||.||), \bar{\gamma})\).

Further, by \(\bar{w} = (w_{nk}) \in \ell_2^{\infty}\), we mean sup \(w_{nk} < \infty\). We denote \(A(\lambda) = \max(1,|\lambda|)\) and the zero element of this class by \(\theta = (\theta, \theta, \theta, \ldots)\).

3 Main Results

In this section, we investigate some conditions in terms of \(\bar{\omega}\) and \(\bar{\gamma}\) so that a class \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) is contained in or equal to another class of the same kind and then explore some of the preliminary results that characterize the linear topological structure of \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) when topologized it with suitable natural paranorm.

As far as the linear space structure of \(\ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) over the field of complex numbers is concerned, we throughout take the coordinatewise operations, i.e., for

\[
\bar{x} = (x_{nk}), \bar{y} = (y_{nk}) \text{ and scalar } \lambda, \bar{x} + \bar{y} = (x_{nk} + y_{nk}) \text{ and } \lambda \bar{x} = (\lambda x_{nk}).
\]

**Theorem 3.1.** For any \(\bar{w}, \ell^2((X,||.||), \bar{\mu}, \bar{\omega}) \subset \ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) if and only if

\[
\lim_{n+k \to \infty} \frac{\gamma_{nk}}{\mu_{nk}} < \infty.
\]

**Proof.** Suppose \(\limsup_{n+k \to \infty} \frac{\gamma_{nk}}{\mu_{nk}} \omega_{nk} < \infty\) and \(\bar{x} \in \ell^2((X,||.||), \bar{\mu}, \bar{\omega})\). Then there exists a constant \(L > 0\) such that \(\gamma_{nk} \omega_{nk} < L|\mu_{nk}|^{w_{nk}}\) for all sufficiently large value of \(n, k\). This means that for all sufficiently large values of \(n, k\),

\[
|\gamma_{nk} x_{nk}||^{w_{nk}} < L|\mu_{nk} x_{nk}|^{w_{nk}}
\]

Thus \(\sum \sum ||\mu_{nk} x_{nk}||^{w_{nk}} < \infty\) implies that \(\sum \sum ||\gamma_{nk} x_{nk}||^{w_{nk}} < \infty\), i.e., \(\bar{x} \in \ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\) and hence \(\ell^2((X,||.||), \bar{\mu}, \bar{\omega}) \subset \ell^2((X,||.||), \bar{\gamma}, \bar{\omega})\).
Conversely, let the inclusion holds but \( \limsup_{n+k \to \infty} \left| \frac{2}{\mu_{nk}} \right|^{w_{nk}} = \infty \). Then there exist subsequence \((n(i))\) of \((n)\) and \((k(i))\) of \((k)\) respectively such that for each \(i \geq 1\),

\[
\| \gamma_{n(i)k(i)} |^{w_{n(i)k(i)}}) > i \| \mu_{n(i)k(i)} |^{w_{n(i)k(i)}} \]

(1)

Thus for \(z_{nk} \in X\) with \(\|z_{nk}\| = 1\), the sequence \(\bar{x} = (x_{nk})\) defined by

\[
x_{nk} = \begin{cases} 
\mu_{nk}^{-1 - 2/w_{nk}}, n = n(i), k = k(i), i \geq 1 \\
\theta, \text{ otherwise}
\end{cases}
\]

(2)

is in \(\ell^2((X, \|\cdot\|), \bar{\mu}, \bar{w})\), since for \(n = n(i), k = k(i), i \geq 1\) and in view of (1) and (2),

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \mu_{nk} x_{nk} \|^{w_{nk}} = \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \mu_{nk} x_{nk} \|^{w_{nk}} = \sum_{i=1}^{\infty} \| \mu_{n(i)k(i)} x_{n(i)k(i)} \|^{w_{n(i)k(i)}} = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.
\]

But \(\bar{x}\) does not belong to \(\ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w})\), since for \(n = n(i), k = k(i), i \geq 1\),

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \mu_{nk} x_{nk} \|^{w_{nk}} = \sum_{i=1}^{\infty} \| \mu_{n(i)k(i)} x_{n(i)k(i)} \|^{w_{n(i)k(i)}} = \sum_{i=1}^{\infty} \frac{|\lambda_{n(i)k(i)}|}{\mu_{n(i)k(i)}} |^{w_{n(i)k(i)}} \frac{1}{i^2} > \sum_{i=1}^{\infty} \frac{1}{i} = \infty,
\]

a contradiction. This completes the proof.

\[\square\]

**Theorem 3.2.** For any \(\bar{w} = (w_{nk}), \ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w}) \subset \ell^2((X, \|\cdot\|), \bar{\mu}, \bar{w})\)

if and only if \(\liminf_{n+k \to \infty} \left| \frac{2}{\mu_{nk}} \right|^{w_{nk}} > 0\).

**Proof.** Suppose \(\liminf_{n+k \to \infty} \left| \frac{2}{\mu_{nk}} \right|^{w_{nk}} > 0\) and \(\bar{x} = (x_{nk}) \in \ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w})\). Then there exists \(I > 0\) such that

\[
I \| \mu_{nk} |^{w_{nk}} < \| \gamma_{nk} |^{w_{nk}} \]

for all sufficiently large values of \(n, k\). Thus

\[
I \| \mu_{nk} x_{nk} \|^{w_{nk}} \leq \| \gamma_{nk} x_{nk} \|^{w_{nk}}
\]

for all sufficiently large values of \(n, k\). From the above inequality and we see that

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \mu_{nk} x_{nk} \|^{w_{nk}} < \infty
\]

i.e., \(\hat{x} \in \ell^2((X, \|\cdot\|), \bar{\mu}, \bar{w})\) and hence

\[
\ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w}) \subset \ell^2((X, \|\cdot\|), \bar{\mu}, \bar{w}).
\]

Conversely, let the inclusion holds but \(\liminf_{n+k \to \infty} \left| \frac{2}{\mu_{nk}} \right|^{w_{nk}} = 0\). Then there exist subsequences \((n(i))\) of \((n)\) and \((k(i))\) of \((k)\) respectively such that for each \(i \geq 1\)

\[
\| \gamma_{n(i)k(i)} |^{w_{n(i)k(i)}} \| \geq \| \mu_{n(i)k(i)} |^{w_{n(i)k(i)}} \|
\]

(3)

For \(z_{nk} \in X\) and \(\|z_{nk}\| = 1\), we define a sequence \(\bar{x} = (x_{nk})\) by

\[
x_{nk} = \begin{cases} 
\gamma_{nk}^{-1 - 2/w_{nk}} x_{nk}, k = k(i), i \geq 1 \\
\theta, \text{ otherwise}
\end{cases}
\]

(4)

Then as proved in Theorem 3.1, for \(n = n(i), k = k(i), i \geq 1\) and in view of (3) and (4), we can prove that \(\bar{x}\) is in \(\ell^2((X, \|\cdot\|), \bar{\gamma}, \bar{w})\) and \(\bar{x} \in \ell^2((X, \|\cdot\|), \bar{\mu}, \bar{w})\), a contradiction. This completes the proof. \[\square\]
On Topological Structure of Total Paranormed Double Sequence Space \((\ell^2((X, ||||), \bar{\gamma}, \bar{w}), G)\)

On combining Theorems 3.1 and 3.2, we get

**Theorem 3.3.** For any \(\bar{w} = (w_{nk}), \ell^2((X, ||||), \bar{\gamma}, \bar{w}) = \ell^2((X, ||||), \bar{\mu}, \bar{w})\) if and only if

\[
0 < \liminf_{n+k \to \infty} \left| \frac{\gamma_{nk}}{\mu_{nk}} \right| w_{nk} \leq \limsup_{n+k \to \infty} \left| \frac{\gamma_{nk}}{\mu_{nk}} \right| w_{nk} < \infty.
\]

**Theorem 3.4.** For any \(\bar{\gamma} = (\gamma_{nk}), w_{nk} \leq v_{nk}\) for all but finitely many \(n, k\), then

\[
\ell^2((X, ||||), \bar{\gamma}, \bar{w}) \subset \ell^2((X, ||||), \bar{\mu}, \bar{v}).
\]

**Proof.** Let \(w_{nk} \leq v_{nk}\) for all finitely many \(n, k\). If \(\bar{x} = (x_{nk}) \in \ell^2((X, ||||), \bar{\gamma}, \bar{w})\) then clearly \(\bar{X} \in \ell^2((X, ||||), \bar{\gamma}, \bar{v})\) because \(\|\gamma_{nk}x_{nk}\| \leq 1\) for all large values of \(n, k\). This completes the proof.

On combining Theorems 3.1 and 3.4, we get

**Theorem 3.5.** For any \(\bar{\gamma} = (\gamma_{nk}), \bar{\mu} = (\mu_{nk}), \bar{w} = (w_{nk})\) and \(\bar{v} = (v_{nk})\) if \(\liminf_{n+k \to \infty} \left| \frac{\gamma_{nk}}{\mu_{nk}} \right| w_{nk} > 0\), and \(w_{nk} \leq v_{nk}\) for all but finitely many \(n, k\), hold together, then

\[
\ell^2((X, ||||), \bar{\gamma}, \bar{w}) \subset \ell^2((X, ||||), \bar{\mu}, \bar{v}).
\]

**Theorem 3.6.** \(\ell^2((X, ||||), \bar{\gamma}, \bar{w})\) forms a linear space over the field of complex numbers \(\mathbb{C}\) if and only if \(\bar{w} = (w_{nk}) \in \ell^2_{\infty}\).

**Proof.** Let \(\bar{w} = (w_{nk}) \in \ell^2_{\infty}\) and \(\bar{x} = (x_{nk}), \bar{y} = (y_{nk}) \in \ell^2((X, ||||), \bar{\gamma}, \bar{w})\) then

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}x_{nk}\| w_{nk} < \infty \quad \text{and} \quad \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}y_{nk}\| w_{nk} < \infty.
\]

Now,

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}(x_{nk} + y_{nk})\| w_{nk} \leq \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}x_{nk}\| w_{nk} + \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}y_{nk}\| w_{nk} < \infty.
\]

Hence \(\bar{x} + \bar{y} \in \ell^2((X, ||||), \bar{\gamma}, \bar{w})\). Also, it is clear that for any scalar \(\lambda\),

\[
\lambda \bar{x} \in \ell^2((X, ||||), \bar{\gamma}, \bar{w}), \quad \text{since} \quad \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\lambda\gamma_{nk}x_{nk}\| w_{nk} = \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} |\lambda| w_{nk} \|\gamma_{nk}x_{nk}\| w_{nk} \leq A(\lambda) \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}x_{nk}\| w_{nk} < \infty.
\]

Conversely, if \(\bar{w} = (w_{nk}) \notin \ell^2_{\infty}\) then there exist subsequences \((n(i))\) of \((n)\) and \((k(i))\) of \((k)\) such that

\[
w_{n(i)k(i)} > 1 \quad \text{for each} \quad i \geq 1. \quad (5)
\]

Now taking \(z_{nk} \in X\) with \(\|z_{nk}\| = 1\), we define a sequence \(\bar{x} = (x_{nk})\) by

\[
x_{nk} = \begin{cases} n^{-1} i^{-2/w_{nk}z_{nk}}, n = n(i), k = k(i), i \geq 1 \text{ and} \\ \theta, \text{otherwise.} \end{cases} \quad (6)
\]

Then for \(n = n(i), k = k(i), i \geq 1\) and in view of \((5)\) and \((6)\), we have

\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk}x_{nk}\| w_{nk} = \sum_{i=1}^{\infty} \|\gamma_{n(i)k(i)}x_{n(i)k(i)}\| w_{n(i)k(i)} = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.
\]

56
This shows that $\bar{x}$ is in $\ell^2((X, \|\cdot\|), \gamma, \bar{w})$. But on the other hand for $n = n(i), k = k(i), i \geq 1$ and for the scalar $\lambda = 4$ we have
\[
\|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk = \|\gamma_{n(i)k(i)}x_{n(i)k(i)}\|_{\bar{w}n(i)k(i)}^nk = |4|^{w(n(i)k(i))} \frac{1}{i^2} > \frac{4^i}{i^2} > 1
\]
for each $i \geq 1$, and therefore
\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk > \infty,
\]
which shows that
\[
\lambda \bar{x} \notin \ell^2((X, \|\cdot\|), \gamma, \bar{w}).
\]
Hence $\ell^2((X, \|\cdot\|), \gamma, \bar{w})$ is a linear space if and only if, $\bar{w} = (w_{nk}) \in \ell^2_\infty$. This completes the proof. \hfill \Box

In the following, let $\bar{w} = (w_{nk}) \in \ell^2_\infty$ and consider $\bar{x} \in \ell^2((X, \|\cdot\|), \gamma, \bar{w})$, we define
\[
G(\bar{x}) = \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk
\]
(7)

**Theorem 3.7.** Let $\bar{w} = (w_{nk}) \in \ell^2_\infty$ for each $n, k \geq 1$ and $X$ be a normed space. Then $(\ell^2((X, \|\cdot\|), \gamma, \bar{w}), G)$ defined by (7) forms a total paranormed space.

**Proof.** For any $\bar{x}, \bar{y} \in \ell^2((X, \|\cdot\|), \gamma, \bar{w})$, it can be easily verified that $G$ satisfy following properties of paranormed space.

Clearly, $G(\bar{x}) \geq 0$ and $G(\bar{x}) = 0$ if and only if $\bar{x} = 0$.

\[
G(\bar{x} + \bar{y}) \leq G(\bar{x}) + G(\bar{y}), \text{ and } G(\lambda \bar{x}) \leq A(\lambda) \cdot G(\bar{x}), \text{ where } \lambda \in \mathbb{C}.
\]

So obviously PN1, PN2 and PN3 follow.

Here we prove the continuity of scalar multiplication, i.e., PN4. For this, it suffices to prove that
(a) if $\bar{x}^{(i)} \to \bar{\theta}$ as $i \to \infty$ and $\lambda_i \to \lambda$ imply $G \left(\lambda_i \bar{x}^{(i)}\right) \to 0$ as $i \to \infty$.
(b) if $\lambda_i \to 0$ as $i \to \infty$ implies $G \left(\lambda_i \bar{x}^{(i)}\right) \to 0$ as $i \to \infty$ for each $\bar{x} \in \ell^2((X, \|\cdot\|), \gamma, \bar{w})$.

Now to prove (a) suppose that $|\lambda_i| \leq L$ for all $i \geq 1$. Then
\[
G \left(\lambda_i \bar{x}^{(i)}\right) \leq \sup_{n,k} |\lambda_i|^{\bar{w}nk} \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk \leq A(\lambda) G \left(\bar{x}^{(i)}\right)
\]
whence (a) follows.

Next if $\bar{x} \in \ell^2((X, \|\cdot\|), \gamma, \bar{w})$ then for $\varepsilon > 0$ there exists $I$ such that
\[
\sum_{n+k \geq I} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk < \frac{\varepsilon}{2}.
\]
Further if $\gamma_i \to 0$, we can find $K$ such that when $i \geq K$, we have
\[
\sum_{2 \leq n+k \leq i-1} |\lambda_i|^{\bar{w}nk} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk < \frac{\varepsilon}{2} \text{ and } |\alpha_i| \leq 1.
\]
Thus
\[
G(\lambda_i \bar{x}) \leq \sum_{2 \leq n+k \leq i-1} \|\lambda_i \gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk + \sum_{n+k \geq 1} \|\gamma_{nk} x_{nk}\|_{\bar{w}nk}^nk < \varepsilon \text{ for all } i \geq K
\]
and hence (b) follows. \hfill \Box

**Theorem 3.8.** Let $\bar{w} = (w_{nk}) \in \ell^2_\infty$ for each $n, k \geq 1$ and $X$ be a Banach space. Then $(\ell^2((X, \|\cdot\|), \gamma, \bar{w}), G)$ is complete with respect to the metric $d(\bar{x}, \bar{y}) \leq Q(\bar{x} - \bar{y})$. 

57
Proof. Let \((\bar{x}^i)\) be a Cauchy sequence in \(\ell^2((X, \| \cdot \|), \bar{\gamma}, \bar{w})\). Thus for \(0 < \varepsilon < 1\), there exists \(K\) such that
\[
G(\bar{x}^i - \bar{x}^f) = \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \gamma_{nk}x^i_{nk} - \gamma_{nk}x^f_{nk} \|^{w_{nk}} < \varepsilon, \text{ for all } i, \ell \geq K.
\]
Hence for each \(n, k \geq 1\)
\[
\| x^i_{nk} - x^f_{nk} \| < |\gamma_{nk}|^{1/\varepsilon} \| \gamma_{nk} \|^{1/\varepsilon} < |\gamma_{nk}|^{-1} \varepsilon, \text{ for all } i, \ell \geq K.
\]
This shows that for each \(n, k, (x^i_{nk})_{i=1}^\infty\) is a Cauchy sequence in \(X\) and because of completeness of \(X, x^i_{nk} \to x_{nk}\) in \(X\), say \(i \to \infty\) for each \(n, k \geq 1\). Being a Cauchy sequence \((\bar{x}^i)\) is bounded, that is there exists an \(L > 0\) such that for all \(i\) and \(K \leq 2,\)
\[
\sum_{2 \leq n+k \leq K} \| \gamma_{nk}x^i_{nk} \|^{w_{nk}} \leq L
\]
. First taking \(i \to \infty\) and then \(N \to \infty\) we easily obtain that
\[
\lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \gamma_{nk}x_{nk} \|^{w_{nk}} \leq L
\]
which implies that \(\bar{x} = (x_{nk}) \in \ell^2((X, \| \cdot \|), \bar{\gamma}, \bar{w})\). Now for any \(K_1\), by (3.8), we have
\[
\sum_{2 \leq n+k \leq K} \| \gamma_{nk}x^i_{nk} - \gamma_{nk}x^f_{nk} \|^{w_{nk}} < \varepsilon, \text{ for all } i, \ell \geq K.
\]
and so letting \(\ell \to \infty\) first and then \(K \to \infty\), we get
\[
G((\bar{x}^{(i)}) - \bar{x}) = \lim_{K \to \infty} \sum_{2 \leq n+k \leq K} \| \gamma_{nk}x^k_{nk} - \gamma_{nk}x_{nk} \|^{w_{nk}} \leq \varepsilon.
\]
This shows that \(\bar{x}^k \to \bar{x}\) in \(\ell^2((X, \| \cdot \|), \bar{\gamma}, \bar{w})\) as \(i \to \infty\). This proves the completeness of \(\ell^2((X, \| \cdot \|), \bar{\gamma}, \bar{w})\).

\[\square\]

4 Conclusion

In this paper, we have explored some conditions that characterize the linear topological structures and containment relations on double sequence space with their terms in a normed space as a generalization of the familiar sequence space. In fact, these results can be used for further generalization to investigate many other properties of the normed spaces, 2-normed spaces, and other vector-valued sequences.

References


