On I-Convergence Difference Sequence Spaces Defined by Orlicz Function in 2-Normed Space

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Abstract: The difference sequence spaces of type I-convergent, I-null, bounded I-convergent, and bounded I-null in 2-normed space are introduced and studied using the Orlicz function. Investigations into the pertinent characteristics of these spaces led us to the establishment of some inclusion relations.

Keywords: Orlicz function, 2-Normed space, Difference sequence space, Ideal convergence

1 Introduction and Preliminaries

In the development of calculus and some other branches of mathematics, infinite sequences and series played an important role. Numerous disciplines of mathematics have been studied using infinite sequences and series. However, the mathematicians were facing the problem of calculating the limits of infinite series, in particular with those having divergent in behavior. Then mathematicians developed the various types of convergence to assign a limit, in some sense, to divergent series and sequences. In this work, we introduce and study the sequence spaces defined by Orlicz function and ideal convergence. We recall the definitions and notations used in this work before moving on to the main results.

Definition 1.1. [9] A function $M : [0, \infty) \rightarrow [0, \infty)$ is said to be an Orlicz function if it is continuous, convex and non decreasing with the conditions $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. An Orlicz function $M$ can be represented in the following integral form

$$M(x) = \int_0^x q(t)dt,$$

where $q$ is known as the kernel of Orlicz function $M$ which is right- differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, $q$ is non decreasing, and $q(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Definition 1.2. [9] An Orlicz function $M$ is said to satisfy $\Delta_2$-condition for all values of $x$, if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$, for all $x \geq 0$. The $\Delta_2$-condition is equivalent to the satisfaction of the inequality $M(Qx) \leq KQM(x)$ for all values of $x$ for which $Q > 1$.

Orlicz sequence spaces are one of the natural generalizations of classical sequence spaces $l_p$. They were first considered by Orlicz in 1936. Afterwards, Lindenstrauss and Tzafriri [10] used Orlicz function in order to construct Orlicz sequence space $l_M$ given by

$$l_M = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}$$

of scalars $(x_k)$. The first detailed study on Orlicz spaces was given by Krasnosel’skii and Rutickii [9]. The Orlicz sequence space $l_M$ becomes a Banach space with the norm given by

$$||x||_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.$$

Moreover, $l_M$ is closely related to the space $l_p$ with $M(x) = x^p; 1 \leq p < \infty$. 


Definition 1.3. [2] Let $X$ be a vector space with $\dim(X) > 1$. A function $\|\cdot, \cdot\|: X \times X \to \mathbb{R}$ satisfying

1. $\|x, y\| \geq 0$, $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent
2. $\|x, y\| = \|y, x\|
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$ for any real $\alpha$
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$

is called a 2-norm on $X$. The pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed space.

The 2-normed space is the generalization concept of normed space.

Geometrically, 2-norm measures the area of the parallelogram spanned by the two vectors.

Definition 1.4. [7] Kizmaz defined the difference sequence spaces by

\[ c_0(\Delta) = \{ x = (x_k) : \Delta x \in c_0 \}, \]
\[ c(\Delta) = \{ x = (x_k) : \Delta x \in c \}, \]
\[ l_\infty(\Delta) = \{ x = (x_k) : \Delta x \in l_\infty \}, \]

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and showed that these spaces are Banach spaces with the norm given by $||x|| = |x_1| + ||(\Delta x)||_\infty$.

The notion of ideal convergence was introduced as a generalization of statistical convergence, first by Kostyrko et al. [8]. For more details about these types of sequence spaces, one may refer to Dutta [1], Ghimire [3], Ghimire and Pahari [4, 5], Hazarika et al. [6], Mursaleen and Alotaibi [11], Mursaleen and Mohiuddine [12], Mursaleen and Sharma [13], Sahiner et al. [15], Salat et al. [16], Savas and Gurdal [17], Savas [18, 19], Tripathy and Hazarika [20] and many others.

Definition 1.5. [8] A family of sets $I \subseteq 2^X$ is said to be an ideal on $X$ if

1. $\emptyset \in I$
2. $A \in I$ and $B \subseteq A$ implies that $B \in I$ (Hereditary Property)
3. $A, B \in I$ implies $A \cup B \in I$ (Additive Property).

Definition 1.6. [8] Let $I \subseteq 2^N$ be non trivial ideal in $\mathbb{N}$. The sequence $(x_n)$ in a normed space $(X, \|\cdot\|)$ is said to be ideal convergent ($I$-convergent) to $x \in X$ if the set

\[ A(\epsilon) = \{ n \in \mathbb{N} : \|x_n - x\| \geq \epsilon \} \in I, \text{ for each } \epsilon > 0. \]

This can be written as

\[ I - \lim_{n \to \infty} x_n = x. \]

Definition 1.7. [20] A sequence space $S$ is said to be sequence algebra if $(x_k), (y_k) = (x_ky_k) \in S$ whenever $(x_k), (y_k) \in S$.

Definition 1.8. [14] A sequence space $X$ is said to be solid (normal) if for any sequence $(x_k)$ in a sequence space $X$ and for all sequences $(\lambda_k)$ of scalars with $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$ implies that $(\lambda_kx_k) \in X$.

Definition 1.9. [15] Let $I \subseteq 2^N$ be a non trivial ideal in $\mathbb{N}$. The sequence $(x_n)$ of 2-normed space $(X, \|\cdot, \cdot\|)$ is called $I$-convergent to $x$ if the set $\{ n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon \} \in I$ for each $z \in X$ and for each $\epsilon > 0$.

This can be written as

\[ I - \lim_{n \to \infty} ||x_n - x, z|| = 0. \]
2 Main Results

In this section, we introduce and investigate the following classes of difference sequence spaces defined by ideal convergence and Orlicz function in 2-normed space by extending the work done in [3].

Let \((X, \|\cdot\|)\) be a 2-normed space. Assign \(\omega\) to represent the space containing all vector-valued sequences defined over \((X, \|\cdot\|)\). We now define the following classes of sequences

\[c^I(M, \Delta, \|\cdot\|) = \left\{ c = (x_k) \in \omega : I - \lim_{k \to \infty} M \left( \frac{\|\Delta x_k - l, z\|}{\rho} \right) = 0, \text{ for some } \rho > 0, l \in X, z \in X \right\}\]

\[c_0^I(M, \Delta, \|\cdot\|) = \left\{ c = (x_k) \in \omega : I - \lim_{k \to \infty} M \left( \frac{\|\Delta x_k, z\|}{\rho} \right) = 0, \text{ for some } \rho > 0, z \in X \right\}\]

\[l_\infty(M, \Delta, \|\cdot\|) = \left\{ x = (x_k) \in \omega : \sup_{k} M \left( \frac{\|\Delta x_k, z\|}{\rho} \right) < \infty, \text{ for some } \rho > 0, z \in X \right\}\]

Also, we define

\[m^I(M, \Delta, \|\cdot\|) = c^I(M, \Delta, \|\cdot\|) \cap l_\infty(M, \Delta, \|\cdot\|)\]

\[m_0^I(M, \Delta, \|\cdot\|) = c_0^I(M, \Delta, \|\cdot\|) \cap l_\infty(M, \Delta, \|\cdot\|)\]

The classes of sequences denoted by \(c^I(M, \Delta, \|\cdot\|), c_0^I(M, \Delta, \|\cdot\|), m^I(M, \Delta, \|\cdot\|)\) and \(m_0^I(M, \Delta, \|\cdot\|)\) are categorized as \(I\)-convergent, \(I\)-null, bounded \(I\)-convergent and bounded \(I\)-null respectively.

**Theorem 2.1.** The classes \(c^I(M, \Delta, \|\cdot\|), c_0^I(M, \Delta, \|\cdot\|), m^I(M, \Delta, \|\cdot\|)\) and \(m_0^I(M, \Delta, \|\cdot\|)\) are linear spaces.

**Proof.** We prove that the class \(c^I(M, \Delta, \|\cdot\|)\) is a linear space. One can show that the remaining classes are linear spaces in the same way.

Let \(x = (x_k)\) and \(y = (y_k)\) be sequences in \(c^I(M, \Delta, \|\cdot\|)\). Then, there exist \(\rho_1, \rho_2 > 0\) such that

\[I - \lim_{k \to \infty} M \left( \frac{\|\Delta x_k - l_1, z\|}{\rho_1} \right) = 0\]

and

\[I - \lim_{k \to \infty} M \left( \frac{\|\Delta y_k - l_2, z\|}{\rho_2} \right) = 0\]

for some \(l_1, l_2 \in X\) and \(z \in X\).

Thus, for given positive \(\epsilon\), we have

\[N_1 = \left\{ k \in \mathbb{N} : M \left( \frac{\|\Delta x_k - l_1, z\|}{\rho_1} \right) > \frac{\epsilon}{2} \right\} \in I\]  \(\text{(1)}\)

\[N_2 = \left\{ k \in \mathbb{N} : M \left( \frac{\|\Delta y_k - l_2, z\|}{\rho_2} \right) > \frac{\epsilon}{2} \right\} \in I\]  \(\text{(2)}\)

Let \(\alpha\) and \(\beta\) be the scalars. Let us choose

\[\rho = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\} .\]

Using non-decreasing and convexity properties of Orlicz function \(M\), we have

\[M \left( \frac{\|\alpha \Delta x_k + \beta \Delta y_k\|}{\rho} \right) \leq M \left( \frac{|\alpha| \|\Delta x_k - l_1, z\|}{\rho_1} + \frac{|\beta| \|\Delta y_k - l_2, z\|}{\rho_2} \right) \]

\[\leq M \left( \frac{\|\Delta x_k - l_1, z\|}{\rho_1} \right) + M \left( \frac{\|\Delta y_k - l_2, z\|}{\rho_2} \right)\]

From (1) and (2), we have

\[\left\{ k \in \mathbb{N} : M \left( \frac{\|\alpha \Delta x_k + \beta \Delta y_k\|}{\rho} \right) > \epsilon \right\} \subset N_1 \cup N_2 .\]

Thus, \(c^I(M, \Delta, \|\cdot\|)\) is a linear space.
Hence, we conclude that $M_1, M_2$ satisfy the $\Delta_2$-condition, then

1. $S(M_1, \Delta, \|\cdot\|) \subseteq S(M_2 \cdot M_1, \Delta, \|\cdot\|)$

2. $S(M_1, \Delta, \|\cdot\|) \cap S(M_2, \Delta, \|\cdot\|) \subseteq S(M_1 + M_2, \Delta, \|\cdot\|)$ for $S = c^1, c^0, m^1, m^0.$

**Proof.** We prove these results for the space $S = c^1_0.$ The proofs for the remaining spaces can be done in the similar way.

1. Let $x = (x_k) \in c^1_0(M_1, \Delta, \|\cdot\|)$. Then there exists a positive number $\rho$ such that
   
   $I - \lim_{k \to \infty} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) = 0, \text{ for } z \in X.

Let $\epsilon > 0.$ Since $M_1$ is an Orlicz function, there exists $0 < \delta < 1$ such that $M_1(u) < \epsilon$, whenever $\delta u \geq 0$. Now we define the following sets

$$
D_1 = \left\{ k \in \mathbb{N} : M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \leq \delta \right\}
$$

$$
D_2 = \left\{ k \in \mathbb{N} : M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) > \delta \right\}
$$

If $k \in D_2$ then

$$
M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) < \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) < 1 + \left\{ \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\}
$$

Since $M_2$ is convex and non-decreasing, we can write

$$
M_2 \left\{ M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} < M_2 \left\{ 1 + \left\{ \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} \right\}
$$

$$
< 1 + \frac{1}{2} M_2(2) + \frac{1}{2} M_2 \left\{ 2 \cdot \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\}
$$

Since $M_2$ satisfies $\Delta_2$-condition, we have

$$
M_2 \left\{ M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} < \frac{1}{2} K \left\{ \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} M_2(2) + \frac{1}{2} K \left\{ \frac{1}{\delta} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} M_2(2)
$$

$$
= \frac{K}{\delta} M_2(2) M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right)
$$

For $k \in D_1$, we have

$$
M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) < \delta \implies M_2 \left\{ M_1 \left( \frac{\|\Delta x_k, z\|}{\rho} \right) \right\} < \epsilon
$$

Hence, we conclude that

$$
c^1_0(M_1, \Delta, \|\cdot\|) \subset c^1_0(M_2 \cdot M_1, \Delta, \|\cdot\|).
$$

2. Let $x = (x_k) \in c^1_0(M_1, \Delta, \|\cdot\|) \cap c^1_0(M_2, \|\cdot\|).$

For positive constants $\rho_1$ and $\rho_2$, we have

$$
I - \lim_{k \to \infty} M_1 \left( \frac{\|\Delta x_k, z\|}{\rho_1} \right) = 0, \text{ for } z \in X.
$$

and

$$
I - \lim_{k \to \infty} M_2 \left( \frac{\|\Delta x_k, z\|}{\rho_2} \right) = 0, \text{ for } z \in X.
$$

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Choose \( \rho = \max\{ \rho_1, \rho_2 \} \). Then,

\[
(M_1 + M_2) \left( \frac{\| \Delta x_k, z \|}{\rho} \right) = M_1 \left( \frac{\| \Delta x_k, z \|}{\rho} \right) + M_2 \left( \frac{\| \Delta x_k, z \|}{\rho} \right) \\
\leq M_1 \left( \frac{\| \Delta x_k, z \|}{\rho_1} \right) + M_2 \left( \frac{\| \Delta x_k, z \|}{\rho_2} \right)
\]

This yields

\[
c_0^1(M_1, \Delta, ||.,||) \cap c_0^1(M_2, \Delta, ||.,||) \subseteq c_0^1(M_1 + M_2, \Delta, ||.,||)
\]

\[
\square
\]

**Theorem 2.3.** Let \( M \) be an Orlicz function. Then we have

\[
c_0^1(M, \Delta, ||.,||) \subset c_1^1(M, \Delta, ||.,||) \subset l^1_\infty(M, \Delta, ||.,||).
\]

**Proof.** The inclusion \( c_0^1(M, \Delta, ||.,||) \subset c_1^1(M, \Delta, ||.,||) \) is obvious.

Let \( x = (x_k) \in c_1^1(M, \Delta, ||.,||) \). There exists \( \rho > 0 \) such that

\[
I - \lim_{k \to \infty} M \left( \frac{\| \Delta x_k - l, z \|}{\rho} \right) = 0
\]

for \( z \in X \) and \( l \in X \).

Now

\[
M \left( \frac{\| \Delta x_k, z \|}{2\rho} \right) \leq \frac{1}{2} M \left( \frac{\| \Delta x_k - l, z \|}{\rho} \right) + \frac{1}{2} M \left( \frac{\| l, z \|}{\rho} \right)
\]

Taking supremum over \( k \) on each side, one can easily obtain that \( x \in l^1_\infty(M, \Delta, ||.,||) \).

Thus , \( c_0^1(M, \Delta, ||.,||) \subset c_1^1(M, \Delta, ||.,||) \subset l^1_\infty(M, \Delta, ||.,||) \).

\[
\square
\]

**Theorem 2.4.** The spaces \( c_1^1(M, \Delta, ||.,||) \) and \( c_0^1(M, \Delta, ||.,||) \) are sequence algebra.

**Proof.** We prove that the space \( c_1^1(M, \Delta, ||.,||) \) is sequence algebra. The result for the space \( c_0^1(M, \Delta, ||.,||) \) can be proved similarly.

Let \( (x_k), (y_k) \in c_0^1(M, \Delta, ||.,||) \). Then there exists positive constants \( \rho_1 \) and \( \rho_2 \) and for \( z \in X \) such that

\[
I - \lim_{k \to \infty} M \left( \frac{\| \Delta x_k, z \|}{\rho_1} \right) = 0
\]

and

\[
I - \lim_{k \to \infty} M \left( \frac{\| \Delta y_k, z \|}{\rho_2} \right) = 0
\]

Choose \( \rho = \rho_1\rho_2 \).

Then one can easily show that

\[
I - \lim_{k \to \infty} M \left( \frac{\| \Delta x_k \Delta y_k, z \|}{\rho} \right) = 0
\]

It follows that \( x_ky_k = (x_k)(y_k) \in c_0^1(M, \Delta, ||.,||) \). This shows that \( c_0^1(M, \Delta, ||.,||) \) is a sequence algebra.

\[
\square
\]

**Theorem 2.5.** The spaces \( c_0^1(M, \Delta, ||.,||) \) and \( m_0^1(M, \Delta, ||.,||) \) are solid.

**Proof.** We prove that the space \( c_0^1(M, \Delta, ||.,||) \) is solid. The result for the space \( m_0^1(M, \Delta, ||.,||) \) can be proved similarly.

Let \( x = (x_k) \in c_0^1(M, \Delta, ||.,||) \). Then there exists a positive constant \( \rho \) such that

\[
I - \lim_{k \to \infty} M \left( \frac{\| \Delta x_k, z \|}{\rho} \right) = 0 \text{ for } z \in X.
\]
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Let \((\alpha_k)\) be a sequence of scalars such that \(|\alpha_k| \leq 1\) for all natural numbers \(k\). Then

\[
M \left( \left\| \frac{\alpha_k \Delta x_k, z}{\rho} \right\| \right) \leq |\alpha_k| M \left( \left\| \frac{\Delta x_k, z}{\rho} \right\| \right) \leq M \left( \left\| \frac{\Delta x_k, z}{\rho} \right\| \right)
\]

This follows that

\[
I = \lim_{k \to \infty} M \left( \left\| \frac{\alpha_k \Delta x_k, z}{\rho} \right\| \right) = 0.
\]

This shows that \(\alpha_k x_k \in c_0^I(M, \Delta, |||, |.|)\) and hence \(c_0^I(M, |||, |.|)\) is solid.

3 Conclusion

In this work, some of the results that characterize the linear topological properties of the difference sequence spaces of type I-convergent, I-null, bounded I-convergent, and bounded I-null in 2-normed space defined by the Orlicz function have been established. Further extensions of these findings to \(n\)-normed space are possible. Additionally, these findings can be used to create sequence spaces of these or similar types that are richer in algebraic and geometrical properties.

References


