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Discontinuity at Fixed Point on Partial S-Metric Spaces

Nabaraj Adhikari

Central Department of Mathematics, Institute of Science and Technology
Tribhuvan University, Kathmandu, Nepal
Correspondence to: Nabaraj Adhikari, Email: nabaraj.adhikari@cdmath.tu.edu.np

Abstract: *This paper examines the idea of partial-S metric spaces, which extend the traditional structure of S-metric spaces. Within this broader framework, we investigate the issue of discontinuity at fixed points—a significant problem originally raised by Rhodes' in the realm of fixed-point theory. We provide sufficient conditions that ensure the existence of fixed points, even when continuity is not assumed. These findings broaden the scope of fixed-point theory, making it applicable to asymmetric topological environments and offering new perspectives on the behavior of discontinuous mappings.*

Keywords: Fixed point, Partial metric, S-Metric, Discontinuity

1 Introduction

The celebrated Banach contraction principle [4] stands as one of the most widely referenced theorems in fixed point theory, the Banach contraction principle establishes that for a self-mapping T defined on a complete metric space (Ω, d), if there exists a constant $0 \leq a < 1$ such that

$$d(T\zeta, T\eta) \leq a\, d(\zeta, \eta) \quad \text{for all } \zeta, \eta \in \Omega,$$

then T possesses exactly one fixed point $\zeta^* \in \Omega$. Moreover, it is well known that any mapping T satisfying this condition is continuous throughout the space Ω. However, R. Kannan [8] showed that there exist contractive mappings that possess fixed points without being continuous.

Theorem 1.1. *[8] Consider a self-map T on a complete metric space (Ω, d). If for all $\zeta, \eta \in \Omega$, the inequality*

$$d(T\zeta, T\eta) \leq b\,(d(\zeta, T\zeta) + d(\eta, T\eta))$$

holds for some constant b satisfying $0 \leq b < \frac{1}{2}$, then T admits exactly one fixed point in Ω.

Although Kannan's contractive mapping is guaranteed to be continuous at its fixed point, Rhoades [13] raised a significant open problem regarding whether such mappings must be continuous everywhere or under what additional conditions continuity holds globally. Does there exist a contractive condition ensuring fixed points without necessitating continuity at them? This question has led to various proposed solutions. Among them, Pant [5] provided a definitive answer by constructing examples of contractive mappings—under a Meir-Keeler type condition [10]—that may fail to be continuous at their fixed points in a metric space (Ω, d).

To address more general settings, Matthews [9] introduced the framework of partial metric spaces, initially aimed at modeling the denotational semantics of data flow networks. In 1994, he extended the classical Banach contraction theorem to apply within complete partial metric spaces. Later, Sedghi, Shobe, and Aliouche [17] (see also [7]) developed the concept of S-metric spaces. Building on both ideas, Simkhah Asila, Sedghi, and Mitrović [18] first formulated the structure of partial S-metric spaces and subsequently obtained existence results for common fixed points of weakly increasing mappings in these spaces.
More recently, using a result from Zamfirescu [20], Ozgur and Tas [11] addressed Rhoades' open problem in the context of S-metric spaces. The authors formulated a contractive condition sufficient to ensure fixed point existence, even when the mapping is not continuous at that point.

Theorem 1.2. *[11] Let (Ω, S) be a complete $S-$metric space. Let $T : \Omega \to \Omega$ be a self-map such that for all $\zeta, \eta \in \Omega$*

(i) there exist a function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\theta(\kappa) < \kappa$ *for each* $\kappa > 0$ *and*

$$S(T\zeta, T\zeta, T\eta) \leq \theta(M_z^S(\zeta, \eta)).$$

(ii) for a given $\epsilon > 0$, *there exist* $\delta(\epsilon) > 0$ *such that* $\epsilon < M_z^S(\zeta, \eta) < \epsilon + \delta(\epsilon)$ *implies* $S(T\zeta, T\zeta, T\eta) \leq \epsilon$.

Then T *has a unique fixed point say* $\xi \in \Omega$. *Moreover,* T *is continuous at* z *if and only if* $\lim_{\zeta \to \xi} M_z^S(\zeta, z) = 0$, *where*

$$M_z^S(\zeta, \eta) = \max\{a\, S(\zeta, \zeta, \eta), \frac{b}{2}[S(\zeta, \zeta, T\zeta) + S(\eta, \eta, T\eta)], \frac{c}{2}[S(\zeta, \zeta, T\eta) + S(\eta, \eta, T\zeta)]\}$$

where $a, b \in [0, 1)$ *and* $c \in [0, \frac{1}{2})$.

2 Preliminaries

Let Ω be a nonempty set. We adopt the following notation: $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^+$ the positive reals, and $\mathbb{N}$ the natural numbers (positive integers). In [9], Matthews introduced the concept of a partial metric, formally defined as follows:

Definition 2.1 ([9]). A mapping $p\colon \Omega \times \Omega \to \mathbb{R}_0^+$ is said to define a *partial metric* on a non-empty set Ω when it satisfies the following axioms for all elements $\zeta, \eta, \xi \in \Omega$:

(PM1) (Identity) $\zeta = \eta$ if and only if $p(\zeta, \zeta) = p(\zeta, \eta) = p(\eta, \eta)$;

(PM2) (Small self-distances) $p(\zeta, \zeta) \leq p(\zeta, \eta)$;

(PM3) (Symmetry) $p(\zeta, \eta) = p(\eta, \zeta)$;

(PM4) (Modified triangle inequality) $p(\zeta, \xi) \leq p(\zeta, \eta) + p(\eta, \xi) - p(\eta, \eta)$.

We call (Ω, p) a *partial metric space* when p is a partial metric defined on the set Ω. It is worth noting that a partial metric reduces to an ordinary metric when $p(\zeta, \zeta) = 0$ for every $\zeta \in \Omega$.
The notion of an S-metric space was introduced by Sedghi, Shobe, and Aliouche in [17] (see also [7]).

Definition 2.2. [17] Let Ω be a non-empty set. A mapping $S : \Omega \times \Omega \times \Omega \to [0, \infty)$ is said to be S-metric if the following conditions are satisfied:

(S1) $S(\zeta, \eta, \xi) = 0 \Leftrightarrow \zeta = \eta = \xi$;

(S2) $S(\zeta, \eta, \xi) \leq S(\zeta, \zeta, \alpha) + S(\eta, \eta, \alpha) + S(\xi, \xi, \alpha)$ for all $\zeta, \eta, \xi, \alpha \in \Omega$.

The pair (Ω, S) in this instance is referred to as the $S-$ metric space.

Lemma 2.3. *[17] Let* (Ω, S) *be an* S*-metric space. Then we have*

(i) $S(\zeta, \zeta, \eta) = S(\eta, \eta, \zeta)$ *for all* $\zeta, \eta \in \Omega$.

(ii) If $S(\zeta, \zeta, \eta) = 0$ *then* $\zeta = \eta$.

The notion of a partial S-metric space was originally proposed by M. Simkhah Asila, Shaban Sedghi, and Zoran D. Mitrović [18], who combined the frameworks of partial metrics and S-metrics to establish common fixed point theorems for weakly increasing mappings.

Definition 2.4 ([18]). A partial S-metric on a non-empty set Ω is a mapping $S_p : \Omega \times \Omega \times \Omega \to [0, \infty)$ when it satisfies the following axioms for all $\zeta, \eta, \xi, \alpha \in \Omega$:

(SP1) $S_p(\zeta, \zeta, \zeta) = S_p(\eta, \eta, \eta) = S_p(\xi, \xi, \xi) = S_p(\zeta, \eta, \xi)$ if and only if $\zeta = \eta = \xi$;

(SP2) $0 \leq S_p(\zeta, \zeta, \zeta) \leq S_p(\zeta, \zeta, \zeta)$;

(SP3) $S_p(\zeta, \eta, \xi) \leq S_p(\zeta, \zeta, \alpha) + S_p(\eta, \eta, \alpha) + S_p(\xi, \xi, \alpha) - 2S_p(\alpha, \alpha, \alpha)$.

The pair (Ω, S_p) is called partial S-metric space.

Note that every S-metric space is also a partial S-metric space. The following example illustrates that the converse may not be true.

Example 2.5. [18] Let $\Omega = [0, \infty)$ and $S_p : \Omega \times \Omega \times \Omega \to [0, \infty)$ defined by $S_p(\zeta, \eta, \xi) = \max\{\zeta, \eta, \xi\}$ is a partial S-metric space but not an S-metric space.

Lemma 2.6. *[18] Let (Ω, S_p) be a partial $S-$metric space. Then we have*

(i) $S_p(\zeta, \zeta, \eta) = S_p(\eta, \eta, \zeta)$ for all $\zeta, \eta \in \Omega$.

(ii) If $S_p(\zeta, \zeta, \eta) = 0$ then $\zeta = \eta$.

Definition 2.7. Let (Ω, S_p) be a partial S-metric space. For $\epsilon > 0$ define

$$B_{S_p}(\zeta, \epsilon) = \{\eta \in \Omega : S_p(\zeta, \zeta, \eta) < \epsilon + S_p(\zeta, \zeta, \zeta)\}$$

is a open ball in (Ω, S_p) center at ζ and radius ϵ. Each partial S-metric S_p on Ω generate a topology τ_{S_p} which has a base the family of open S_p-balls $\{B_{S_P}(\zeta, \epsilon) : \zeta \in \Omega, \epsilon > 0\}$.

Definition 2.8 ([18])**.** Let (Ω, S_p) be a partial S-metric space and $\{\zeta_n\}$ be a sequence in Ω. Then

(i) $\{\zeta_n\}$ converges to a point $\zeta \in \Omega$ (denoted by $\zeta_n \to \zeta$ as n$\to \infty$) if

$$S_p(\zeta, \zeta, \zeta) = \lim_{n\to\infty} S_p(\zeta_n, \zeta_n, \zeta) = \lim_{n\to\infty} S_p(\zeta_n, \zeta_n, \zeta_n).$$

(ii) $\{\zeta_n\}$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} S_p(\zeta_n, \zeta_n, \zeta_m)$ exists (and is finite). That is,

$$\lim_{n,m\to\infty} S_p(\zeta_n, \zeta_n, \zeta_m) = S_p(\zeta, \zeta, \zeta).$$

(iii) A partial S-metric space (Ω, S_p) is said to be complete if every Cauchy sequence converges with respect to τ_{S_p} to $\zeta \in \Omega$.

Remark 2.9. If $\zeta_n \to \zeta$ as $n \to \infty$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S_p(\zeta_n, \zeta_n, \zeta) - S_p(\zeta, \zeta, \zeta)| < \epsilon$$

and

$$|S_p(\zeta_n, \zeta_n, \zeta_n) - S_p(\zeta, \zeta, \zeta)| < \epsilon$$

for all $n \geq n_0$. Hence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S_p(\zeta_n, \zeta_n, \zeta_n) - S_p(\zeta_n, \zeta_n, \zeta)| < \epsilon$$

for all $n \geq n_0$.

Lemma 2.10. *[18] Let (Ω, S_p) be a partial S-metric space. Suppose sequence $\{\zeta_n\}$ sequence in Ω. Then*

(i) $\{\zeta_n\}$ converges to ζ and, ζ is unique.

(ii) each convergent sequence $\{\zeta_n\}$ is a Cauchy sequence.

S. Sedghi, *et al.*[16] was first introduced to the idea of S_b-metric spaces, which is defined as follows:

Definition 2.11. [16] Let Ω be a non-empty set and $b \geq 1$ is a real number. A function $S_b : \Omega \times \Omega \times \Omega \to [0, \infty)$ is said to be S_b- metric with parameter b if for all $\zeta, \eta, \xi, \alpha \in \Omega$ the following conditions are satisfied:

(Sb1) $S_b(\zeta, \eta, \xi) = 0 \Leftrightarrow \zeta = \eta = \xi$;

(Sb2) $S_b(\zeta, \zeta, \eta) = S_b(\eta, \eta, \zeta)$;

(Sb3) $S_b(\zeta,\eta,\xi) \le b[S_b(\zeta,\zeta,\alpha)+S_b(\eta,\eta,\alpha)+S_b(\xi,\xi,\alpha)]$.

The pair (Ω, S_b) in this instance is referred to as the S_b- metric space.

According to the following lemma, the S_b-metric (for $b=2$)is generated by partial S-metric.

Lemma 2.12. *[18] If (Ω, S_p) is a partial $S-$metric space, then $S^s : \Omega\times\Omega\times\Omega \to [0,\infty)$ defined by*

$$S^s(\zeta,\eta,\xi) = S_p(\zeta,\zeta,\eta)+s_p(\eta,\eta,\xi)+S_p(\xi,\xi,\zeta)-S_p(\zeta,\zeta,\zeta)-S_p(\eta,\eta,\eta)-S_p(\xi,\xi,\xi) \tag{2.1}$$

is an S_b-metric on Ω, with parameter $b=2$.

Lemma 2.13 ([18]). *Let (Ω, S_p) be partial a metric space and S^s be the respective S_b-metric introduced in Lemma 2.12. Then*

(i) the sequence $\{\zeta_n\}$ is a Cauchy in (Ω, S_p) if and only if it is Cauchy in (Ω, S^s).

(ii) (Ω, S_p) is complete if and only if (Ω, S^s) is complete .

(iii) $\lim_{n\to\infty} S^s(\zeta_n,\zeta_n,\zeta)=0$ if and only if $S_p(\zeta,\zeta,\zeta)=\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,\zeta)=\lim_{n,m\to\infty} S_p(\zeta_n,\zeta_n,\zeta_m)$.

Lemma 2.14. *[18] Let $\{\zeta_n\}$ and $\{\eta_n\}$ be two sequences convergent to $\zeta\in\Omega$ and $\eta\in\Omega$, respectively, in a partial S_p-metric space (Ω, S_p). Then*

$$S_p(\zeta,\zeta,\eta)=\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,\eta_n).$$

In particular, $S_p(\zeta,\zeta,\eta)=\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,\eta)$ for all $\eta\in\Omega$.

By virtue of a result by Zamfirescu presented in [20], Nihal Ozgur and Nihal Tas [11] found a solution to Rhoades' open problem regarding the existence of a contractive condition strong enough to ensure a fixed point without requiring the mapping to be continuous at that point within the framework of S-metric spaces. Moreover, recently we studied interpolative contraction and discontinuity at fixed points in partial metric spaces (see details in [1]). In this paper, we extend the results of [11] to the context of partial S-metric spaces.

3 Main Results

Consider a partial S-metric space (Ω, d) and $T:\Omega\to\Omega$ be self map. The number defined as in this section is indicated by

$$M_\xi^{S_p}(\zeta,\eta)=\max\{a\ S_p(\zeta,\zeta,\eta), \frac{b}{2}[S_p(\zeta,\zeta,T\zeta)+S_p(\eta,\eta,T\eta)], \frac{c}{2}[S_p(\zeta,\zeta,T\eta)+S_p(\eta,\eta,T\zeta)]\}$$

where $a,b\in[0,1)$ and $c\in[0,\frac{1}{2})$.

Theorem 3.1. *Let (Ω, S_p) be a complete partial S-metric space. Let $T:\Omega\to\Omega$ be a self-map such that for all $\zeta,\eta\in\Omega$*

(i) there exist a function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ such that $\theta(\kappa)<\kappa$ for each $\kappa>0$ and

$$S_p(T\zeta,T\zeta,T\eta)\le\theta(M_\xi^{S_p}(\zeta,\eta))$$

(ii) for a given $\epsilon>0$, there exist $\delta(\epsilon)>0$ such that $\epsilon<M_\xi^{S_p}(\zeta,\eta)<\epsilon+\delta(\epsilon)$ implies $S_p(T\zeta,T\zeta,T\eta)\le\epsilon$.

Then T has a unique fixed point say $u\in\Omega$. Moreover T is continuous at u if and only if

$$\lim_{\zeta\to u} M_\xi^{S_p}(\zeta,\xi)=S_p(u,u,u).$$

Proof. Under assumption (i), there exists a mapping $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(\kappa) < \kappa$ for each $\kappa > 0$ and

$$S_p(T\zeta, T\zeta, T\eta) \le \theta(M_\xi^{S_p}(\zeta, \eta))$$

for all $\zeta, \eta \in \Omega$. Using the property of θ, we have

$$S_p(T\zeta, T\zeta, T\eta) \le M_\xi^{S_p}(\zeta, \eta) \tag{3.1}$$

where $M_\xi^{S_p}(\zeta, \eta) > 0$. We define $\beta = \max\{a, \frac{b}{2-b}, \frac{c}{2-2c}\}$. Then $\beta < 1$. Let $\zeta_0 \in \Omega$. Define

$$\zeta_{n+1} = T\zeta_n = T^n\zeta_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If there exist n such that $\zeta_{n+1} = \zeta_n$ then ζ_n is a fixed point of T and result is proved. Suppose $\zeta_{n+1} \neq \zeta_n$ for all $n \in \mathbb{N} \cup \{0\}$. Applying condition (i) along with the inequality (3.1), we derive

$$\begin{aligned} S_p(\zeta_n, \zeta_n, \zeta_{n+1}) &= S_p(T\zeta_{n-1}, T\zeta_{n-1}, T\zeta_n) \\ &\le \theta(M_\xi^{S_p}(\zeta_{n-1}, \zeta_n)) \\ &< M_\xi^{S_p}(\zeta_{n-1}, \zeta_n) \\ &= \max\{a\ S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), \frac{b}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, T\zeta_{n-1}) + S_p(\zeta_n, \zeta_n, T\zeta_n)], \\ &\frac{c}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, T\zeta_n) + S_p(\zeta_n, \zeta_n, T\zeta_{n-1})]\} \\ &= \max\{a\ S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n), \frac{b}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + S_p(\zeta_n, \zeta_n, \zeta_{n+1})], \\ &\frac{c}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + S_p(\zeta_n, \zeta_n, \zeta_n)]\} \end{aligned}$$

If $M_\xi^{S_p}(\zeta_{n-1}, \zeta_n) = a\ S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)$. So, we have

$$\begin{aligned} S_p(\zeta_n, \zeta_n, \zeta_{n+1}) &< aS_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \\ &\le \beta S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \\ &< S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n). \end{aligned}$$

Therefore,

$$S_p(\zeta_n, \zeta_n, \zeta_{n+1}) < S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n). \tag{3.2}$$

If $M_\xi^{S_p}(\zeta_{n-1}, \zeta_n) = \frac{b}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + S_p(\zeta_n, \zeta_n, \zeta_{n+1})]$, then

$$\begin{aligned} S_p(\zeta_n, \zeta_n, \zeta_{n+1}) &< \frac{b}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) + S_p(\zeta_n, \zeta_n, \zeta_{n+1})] \\ \implies S_p(\zeta_n, \zeta_n, \zeta_{n+1}) &< \frac{b}{2-b} S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \\ &\le \beta S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \\ &< S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n). \end{aligned}$$

Hence,

$$S_p(\zeta_n, \zeta_n, \zeta_{n+1}) < S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n). \tag{3.3}$$

If $M_\xi^{S_p}(\zeta_{n-1}, \zeta_n) = \frac{c}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + S_p(\zeta_n, \zeta_n, \zeta_n)]$, then

$$\begin{aligned} S_p(\zeta_n, \zeta_n, \zeta_{n+1}) &< \frac{c}{2}[S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_{n+1}) + S_p(\zeta_n, \zeta_n, \zeta_n)] \\ &= \frac{c}{2}[S_p(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n-1}) + S_p(\zeta_n, \zeta_n, \zeta_n)] \quad [\because \text{ Lemma 2.6}] \\ &\le \frac{c}{2}[S_p(\zeta_{n+1}, \zeta_{n+1}, \zeta_n) + S_p(\zeta_{n+1}, \zeta_{n+1}, \zeta_n) + S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n) \\ &- 2S_p(\zeta_n, \zeta_n, \zeta_n) + S_p(\zeta_n, \zeta_n, \zeta_n)] \quad [\because \text{by using (SP2)}] \\ &\le \frac{c}{2}[2S_p(\zeta_{n+1}, \zeta_{n+1}, \zeta_n) + S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)] \\ &= \frac{c}{2}[2S_p(\zeta_n, \zeta_n, \zeta_{n+1}) + S_p(\zeta_{n-1}, \zeta_{n-1}, \zeta_n)] \quad [\because \text{Lemma 2.6}] \end{aligned}$$

$$\begin{aligned} S_p(\zeta_n,\zeta_n,\zeta_{n+1}) &< \frac{c}{2-2c} S_P(\zeta_{n-1},\zeta_{n-1},\zeta_n) \\ &\le \beta S_p(\zeta_{n-1},\zeta_{n-1},\zeta_n) \\ &< S_p(\zeta_{n-1},\zeta_{n-1},\zeta_n). \end{aligned}$$

Hence,

$$S_p(\zeta_n,\zeta_n,\zeta_{n+1}) < S_p(\zeta_{n-1},\zeta_{n-1},\zeta_n). \tag{3.4}$$

Set $q_n = S_p(\zeta_n,\zeta_n,\zeta_{n+1})$, then by the Inequalities (3.2), (3.3) and (3.4) we have

$$q_n < q_{n-1} \tag{3.5}$$

Consequently, the sequence $\{q_n\}$ of positive real numbers is strictly decreasing and has 0 as its greatest lower bound, whereby

$$\lim_{n\to\infty} q_n = q.$$

If possible, let $q > 0$, there exist $k \in \mathbb{N}^+$ such that for all $n \ge k$ implies

$$q < q_n < q + \delta(q). \tag{3.6}$$

From condition (ii) and $q_{n-1} < q_n$ combined with $q_n \le q$ $(\forall n \ge k)$ contradicts inequality (3.6). Hence $q = 0$. That is,

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} S_p(\zeta_n,\zeta_n,\zeta_{n+1}) = 0.$$

Our goal is to show that $\{\zeta_n\}$ is a Cauchy sequence in (Ω, S_p). By using Lemma 2.13, this reduces to proving $\{\zeta_n\}$ is Cauchy in (Ω, S^s). Let $\epsilon > 0$ be given. From inequalities (3.2), (3.3) and (3.4) we have

$$S_p(\zeta_n,\zeta_n,\zeta_{n+1}) < \beta S_p(\zeta_{n-1},\zeta_{n-1,},\zeta_n) < \beta^n S_p(\zeta_0,\zeta_0,\zeta_1). \tag{3.7}$$

Therefore, by Lemma 2.6

$$\begin{aligned} 0 \le S^s(\zeta_n,\zeta_n,\zeta_{n+1}) &= S_p(\zeta_n,\zeta_n,\zeta_n) + s_p(\zeta_n,\zeta_n,\zeta_{n+1}) + S_p(\zeta_{n+1},\zeta_{n+1},\zeta_n) - S_p(\zeta_n,\zeta_n,\zeta_n) \\ &\quad - S_p(\zeta_n,\zeta_n,\zeta_n) - S_p(\zeta_{n+1},\zeta_{n+1},\zeta_{n+1}) \\ &= S_p(\zeta_{n+1},\zeta_{n+1},\zeta_n) + S_p(\zeta_n,\zeta_n,\zeta_{n+1}) - S_p(\zeta_{n+1},\zeta_{n+1},\zeta_{n+1}) \\ &\quad - S_p(\zeta_n,\zeta_n,\zeta_n) \\ &= S_p(\zeta_{n+1},\zeta_{n+1},\zeta_n) + S_p(\zeta_{n+1},\zeta_{n+1},\zeta_n) - S_p(\zeta_{n+1},\zeta_{n+1},\zeta_{n+1}) \\ &\quad - S_p(\zeta_n,\zeta_n,\zeta_n) \ \ [\because \text{Lemma 2.6}] \\ &\le 2S_p(\zeta_{n+1},\zeta_{n+1},\zeta_n) \\ &< 2\beta^n S_p(\zeta_0,\zeta_0,\zeta_1) \ \ \forall \ n \in \mathbb{N}\cup\{0\} \\ \implies \lim_{n\to\infty} S^s(\zeta_n,\zeta_n,\zeta_{n+1}) &< 2 \lim_{n\to\infty} \beta^n S_p(\zeta_0,\zeta_0,\zeta_1) = 0. \end{aligned}$$

By triangle inequality in S_b-metric with $b = 2$, for $m > n$ we have

$$\begin{aligned} S^s(\zeta_n,\zeta_n,\zeta_m) &\le 2[S^s(\zeta_n,\zeta_n,\zeta_{n+1}) + S^s(\zeta_n,\zeta_n,\zeta_{n+1}) + S^s(\zeta_m,\zeta_m,\zeta_{n+1})] \\ &= 2[2S^s(\zeta_n,\zeta_n,\zeta_{n+1}) + S^s(\zeta_{n+1},\zeta_{n+1},\zeta_m)] \\ &\le 2[2S^s(\zeta_n,\zeta_n,\zeta_{n+1}) + 2^2 S^s(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2}) + \cdots + 2^{m-n} S^s(\zeta_{m-1},\zeta_{m-1},\zeta_m)] \\ &= 2^2 S^s(\zeta_n,\zeta_n,\zeta_{n+1}) + 2^3 S^s(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2}) + \cdots + 2^{m-n+1} S^s(\zeta_{m-1},\zeta_{m-1},\zeta_m)] \\ &< 2^3\beta^n S_p(\zeta_0,\zeta_0,\zeta_1) + 2^4\beta^{n+1} S_p(\zeta_0,\zeta_0,\zeta_1) + \cdots + 2^{m-n+2}\beta^{m-1} S_p(\zeta_0,\zeta_0,\zeta_1) \\ &\le 2^3\beta^n[1 + 2\beta + 2^2\beta^2 + \cdots] S_p(\zeta_0,\zeta_0,\zeta_1) \\ &= \frac{2^3\beta^n}{1-2\beta} S_p(\zeta_0,\zeta_0,\zeta_1) \end{aligned}$$

$$\therefore \lim_{m,n\to\infty} S^s(\zeta_n,\zeta_n,\zeta_m) = 0.$$

Hence, $\{\zeta_n\}$ is Cauchy sequence in S_b-metric space (Ω, S^s). Since (Ω, S_p) is complete, then from Lemma 2.13, the sequence $\{\zeta_n\}$ converges in the S_b-metric space (Ω, S^s). Hence, $\lim_{n\to\infty} S^s(\zeta_n, \zeta_n, u) = 0$ for some $u \in \Omega$. So, $S_p(u, u, u) = \lim_{n\to\infty} S_p(\zeta_n, \zeta_n, u) = \lim_{n,m\to} S_p(\zeta_n, \zeta_n, \zeta_m)$. That is,

$$S_p(u, u, u) = \lim_{n\to\infty} S_p(\zeta_{n+1}, \zeta_{n+1}, u) = \lim_{n\to\infty} S_p(T\zeta_n, T\zeta_n, u).$$

We will show that u is a fixed point of T. Suppose, on the contrary u is not fixed point of T. Then using the condition (i), the property of θ, and Lemma 2.6, we have

$$\begin{aligned}
S_p(Tu, Tu, T\zeta_n) &\leq \theta(M_\xi^{S_p}(u, \zeta_n)) < M_\xi^{S_p}(u, \zeta_n) \\
&= \max\{aS_p(u, u, \zeta_n), \frac{b}{2}[S_p(u, u, Tu) + S_p(\zeta_n, \zeta_n, T\zeta_n)], \\
&\frac{c}{2}[S_p(u, u, T\zeta_n) + S_p(\zeta_n, \zeta_n, Tu)]\} \\
&= \max\{aS_p(\zeta_n, \zeta_n, u), \frac{b}{2}[S_p(Tu, Tu, u) + S_p(\zeta_n, \zeta_n, \zeta_{n+1})], \\
&\frac{c}{2}[S_p(T\zeta_n, T\zeta_n, u) + S_p(Tu, Tu, \zeta_n)]\} \\
\implies \lim_{n\to\infty} S_p(Tu, Tu, T\zeta_n) &< \max\{a \lim_{n\to\infty} S_p(\zeta_n, \zeta_n, u), \frac{b}{2}[S_p(Tu, Tu, u) + \lim_{n\to\infty} S_p(\zeta_n, \zeta_n, \zeta_{n+1})], \\
&\frac{c}{2}[\lim_{n\to\infty} S_p(T\zeta_n, T\zeta_n, u) + \lim_{n\to\infty} S_p(Tu, Tu, \zeta_n)]\} \\
\implies S_p(Tu, Tu, u) &< \max\{aS_p(u, u, u), \frac{b}{2}[S_p(Tu, Tu, u) + S_p(u, u, u)], \\
&\frac{c}{2}[S_p(Tu, Tu, u) + S_p(Tu, Tu, u)]\} \\
&\leq \max\{aS_p(u, u, u), bS_p(Tu, Tu, u), cS_p(Tu, Tu, u)\} \\
&< S_p(Tu, Tu, u).
\end{aligned}$$

which is contradiction. So u is a fixed point of T. That is, $Tu = u$.

Uniqueness: Let v be another fixed point of T such that $u \neq v$. Then from condition (i) and Lemma 2.6, we have

$$\begin{aligned}
S_p(u, u, v) &= S_p(Tu, Tu, Tv) \\
&\leq \theta(M_\xi^{S_p}(u, v)) \\
&< M_\xi^{S_p}(u, v) \\
&= \max\{aS_p(u, u, v), \frac{b}{2}[S_p(u, u, Tu) + S_p(v, v, Tv)], \frac{c}{2}[S_p(u, u, Tv) + S_p(v, v, Tu)]\} \\
&= \max\{aS_p(u, u, v), \frac{b}{2}[S_p(u, u, u) + S_p(v, v, v)], \frac{c}{2}[S_p(u, u, v) + S_p(v, v, u)]\} \\
&= \max\{aS_p(u, u, v), \frac{b}{2}[S_p(u, u, u) + S_p(v, v, v)], \frac{c}{2}[S_p(u, u, v) + S_p(u, u, v)]\} \\
&\leq \max\{aS_p(u, u, v), \frac{b}{2}[S_p(u, u, v) + S_p(u, u, v)], cS_p(u, u, v)\} \\
&< S_p(u, u, v)
\end{aligned}$$

which is a contradiction. So, $u = v$. Hence, fixed point of T is unique.
Suppose T is continuous at fixed point u and $\lim_{n\to\infty} S_p(\zeta_n, \zeta_n, u) = S_p(u, u, u)$. Then, we have

$$\lim_{n\to\infty} S_p(T\zeta_n, T\zeta_n, Tu) = S_p(Tu, Tu, Tu) = S_p(u, u, u).$$

Using condition (SP3), we have

$$\begin{aligned} &0 \le S_p(\zeta_n,\zeta_n,T\zeta_n) \le S_p(\zeta_n,\zeta_n,u)+S_p(\zeta_n,\zeta_n,u)+S_p(T\zeta_n,T\zeta_n,u)-2S_p(u,u,u)\\ \implies &0 \le \lim_{n\to\infty} S_p(\zeta_n,\zeta_n,T\zeta_n) \le \lim_{n\to\infty} S_p(\zeta_n,\zeta_n,u)+\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,u)+\lim_{n\to\infty} S_p(T\zeta_n,T\zeta_n,u)\\ &\qquad -2S_p(u,u,u)\\ &\qquad = S_p(u,u,u)+S_p(u,u,u)+S_p(Tu,Tu,u)-2S_p(u,u,u)\\ &\qquad = S_p(u,u,u). \end{aligned}$$

Now,

$$\begin{aligned} \lim_{\zeta_n\to u} M_\xi^{S_p}(\zeta_n,u) =& \max\{a\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,u), \frac{b}{2}[\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,T\zeta_n)+S_p(u,u,Tu)],\\ &\frac{c}{2}[\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,Tu)+\lim_{n\to\infty} S_p(u,u,T\zeta_n)]\}\\ =& \max\{aS_p(u,u,u), bS_p(u,u,u), cS_p(u,u,u)\}\\ =& S_p(u,u,u). \end{aligned}$$

Conversely, suppose $\lim_{\zeta_n\to u} M_\xi^{S_p}(\zeta_n,u) = S_p(u,u,u)$. Then, we have $\lim_{n\to\infty} S_p(\zeta_n,\zeta_n,u) = S_p(u,u,u)$. So

$$\begin{aligned} \lim_{n\to\infty} S_p(T\zeta_n,T\zeta_n,u) &= \lim_{n\to\infty} S_p(\zeta_{n+1},\zeta_{n+1},u)\\ &= S_p(u,u,u)\\ &= S_p(Tu,Tu,u). \end{aligned}$$

Hence T is continuous at fixed point u.

□

Example 3.2. Let $\Omega = [0,1]$ be equipped with the partial S-metric S_p defined by

$$S_p(\zeta,\eta,\omega) = |\zeta-\eta|+|\zeta-\omega|+|\eta-\omega|.$$

Here, (Ω, S_p) is a complete partial S-metric space. Define the self-map $T:\Omega\to\Omega$ by

$$T\zeta = \frac{\zeta}{2}.$$

Let $\theta:\mathbb{R}^+\to\mathbb{R}^+$ be given by $\theta(\kappa)=\frac{\kappa}{2}$. Clearly, $\theta(\kappa)<\kappa$ for all $\kappa>0$. Define

$$M_\xi^{S_p}(\zeta,\eta) = \max\{S_p(\zeta,\zeta,\eta), S_p(\zeta,\zeta,T\zeta), S_p(\eta,\eta,T\eta)\}.$$

So,

$$M_\xi^{S_p}(\zeta,\eta) = \max\{2|\zeta-\eta|, |\zeta|, |\eta|\}.$$

Verification of conditions

(i) We verify

$$S_p(T\zeta,T\zeta,T\eta) = 2\left|\frac{\zeta}{2}-\frac{\eta}{2}\right| = |\zeta-\eta|.$$

On the other hand,

$$\theta(M_\xi^{S_p}(\zeta,\eta)) = \frac{1}{2}\max\{2|\zeta-\eta|, |\zeta|, |\eta|\} \ge |\zeta-\eta|.$$

Thus, $S_p(T\zeta,T\zeta,T\eta) \le \theta(M_\xi^{S_p}(\zeta,\eta))$, satisfying condition (i).

(ii) For any $\epsilon>0$, choose $\delta(\epsilon)=\epsilon$. Suppose

$$\epsilon < M_\xi^{S_p}(\zeta,\eta) < \epsilon+\delta(\epsilon) = 2\epsilon.$$

Then

$$S_p(T\zeta,T\zeta,T\eta) = |\zeta-\eta| \le \frac{1}{2}M_\xi^{S_p}(\zeta,\eta) < \epsilon,$$

satisfying condition (ii).

Fixed Point and continuity

The mapping T has a unique fixed point $u = 0$, since

$$T(0) = \frac{0}{2} = 0.$$

For continuity at $u = 0$, we have

$$\lim_{\zeta \to 0} M_{\xi}^{S_p}(\zeta, 0) = \lim_{\zeta \to 0} \max\{2|\zeta|, |\zeta|, 0\} = 0 = S_p(0, 0, 0).$$

Thus, T is continuous at 0, and the limit condition holds.
This example satisfies all the hypotheses of the theorem, with $u = 0$ as the unique fixed point of T.

Corollary 3.3. *Let (Ω, S_p) be a complete partial $S-$metric space. Let $T : \Omega \to \Omega$ be a self-map such that for all $\zeta, \eta \in \Omega$. If*

(i) $S_p(T\zeta, T\zeta, T\eta) < M_{\xi}^{S_p}(\zeta, \eta)$ with $M_{\xi}^{S_p}(\zeta, \eta) > 0$.

(ii) for a given $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ such that $\epsilon < M_{\xi}^{S_p}(\zeta, \eta) < \epsilon + \delta(\epsilon)$ implies $S_p(T\zeta, T\zeta, T\eta) \leq \epsilon$.

Then T has a unique fixed point say $u \in \Omega$. Moreover, T is continuous at u if and only if

$$\lim_{\zeta \to u} M_{\xi}^{S_p}(\zeta, \xi) = S_p(u, u, u).$$

Proof. Using the similar technique of the Theorem 3.1. □

The following theorem demonstrates that the power contraction of the type $M_{\xi}^{S_p}(\zeta, \eta)$ allows for discontinuity at the fixed point. Denote

$$N_{\xi}^{S_p}(\zeta, \eta) = \max\{a\ S_p(\zeta, \zeta, \eta), \frac{b}{2}[S_p(\zeta, \zeta, T^m\zeta) + S_p(\eta, \eta, T^m\eta)], \frac{c}{2}[S_p(\zeta, \zeta, T^m\eta) + S_p(\eta, \eta, T^m\zeta)]\}$$

where $a, b \in [0, 1)$ and $c \in [0, \frac{1}{2})$.

Theorem 3.4. *Let (Ω, S_p) be a complete partial $S-$metric space. Let $T : \Omega \to \Omega$ be a self-map such that for all $\zeta, \eta \in \Omega$.*

(i) there exist a function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(\kappa) < \kappa$ for each $\kappa > 0$ and

$$S_p(T^m\zeta, T^m\zeta, T^m\zeta) \leq \theta(N_{\xi}^{S_p}(\zeta, \eta)).$$

(ii) for a given $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ such that $\epsilon < N_{\xi}^{S_p}(\zeta, \eta) < \epsilon + \delta(\epsilon)$ implies

$$S_p(T^m\zeta, T^m\zeta, T^m\eta) \leq \epsilon.$$

Then T has a unique fixed point say $u \in \Omega$. Moreover T is continuous at u if and only if

$$\lim_{\zeta \to u} N_{\xi}^{S_p}(\zeta, \xi) = S_p(u, u, u).$$

Proof. By Theorem 3.1, the function T^m has a unique fixed point u. Hence, we have $Tu = TT^m u = T^m(Tu)$ and so Tu is another fixed point of T^m. From the uniqueness of the fixed point we obtain $Tu = u$. That is T has unique fixed point u. □

4 Conclusion

In this study, we have explored the concept of partial-S metric spaces as a natural generalization of classical S-metric spaces. Within this extended framework, we addressed the significant issue of discontinuity at fixed points, a question originally raised by Rhodes in the context of fixed-point theory. By establishing sufficient conditions for the existence of fixed points without requiring continuity, we have broadened the scope of fixed-point theory to include a wider class of mappings and topological structures. These findings contribute to a deeper understanding of fixed-point phenomena in asymmetric settings and open up new avenues for further investigation into discontinuous and non-self mappings in generalized metric spaces.

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References

[1] Adhikari, N. 2025, Interpolative contraction and discontinuity at fixed point on partial metric spaces. *Nepal Journal of Mathematical Sciences*, 6(1), 51–60.

[2] Ahuja, R. K., Magnati, T. L., and Orlin, J. B., 1993, *Network flows: theory, algorithms and applications*, Prentice Hall, Englewood Cliffs, New Jersey.

[3] Areibi, S., and Yang, Z., 2004, Effective memetic algorithms for VLSI design = Genetic algorithms + Local search + Multi-level clustering, *Evolutionary Computation*, 12(3), 327-353.

[4] Banach, S., 1922, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, 3, 133-181.

[5] Bisht, R. K., and Pant, R. P., 2017, Contractive definitions and discontinuity at fixed point, *Appl. Gen. Topol.*, 18, 173-182.

[6] Borah, M., Owens, R. M., and Irwin, M. J., 1995, Transistor sizing for minimizing power consumption of CMOS circuits under delay constraint, *International Symposium on Low Power Design*, 167-172.

[7] Dosenovic, T., Radenovic, S., and Sedghi, S., 2018, Generalized metric spaces: Survey, *TWMS J. Pure Appl. Math.*, 9(1), 3-17.

[8] Kannan, R., 1968, Some results on fixed points, *Bull. Calcutta Math. Soc.*, 62, 71-76.

[9] Matthews, S. G., 1994, Partial metric topology, *Ann. New York Acad. Sci.*, 728, 183-197.

[10] Meir, A., and Keeler, E., 1969, A theorem on contraction mappings, *J. Math. Anal. Appl.*, 28, 326-329.

[11] Ozgur, N., and Tas, N., 2021, A new solution to the Rhoades' open problem with an application, *Acta Univ. Sapientiae, Mathematica*, 13(2), 427-441.

[12] Pant, R. P., 1999, Discontinuity and fixed points, *J. Math. Anal. Appl.*, 240, 284-289.

[13] Rhoades, B. E., 1988, Contractive definitions and continuity, *Contemp. Math.*, 72, 233-245.

[14] Rhoades, B. E., 1977, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, 226, 257-290.

[15] Sedghi, S., and Dung, N. V., 2014, Fixed point theorems on S-metric spaces, *Mat. Vesnik*, 66, 113–124.

[16] Sedghi, S., Gholidahneh, A., Dosenovic, T., Esfahani, J., and Radenovic, S., 2016, Common fixed point of four maps in S_b-metric spaces, *J. Linear Topol. Algebra*, 5(2), 93-104.

[17] Sedghi, S., Shobe, N., and Aliouche, A., 2012, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik*, 64(3), 258-266.

[18] Simkhah Asil, M., Sedghi, S., and Mitrovic, Z. D., 2019, Partial S-Metric Spaces and Coincidence Points, *Filomat*, 33(14), 4613-4626.

[19] Souayah, N., and Mlaiki, N., 2016, A fixed point theorem in S_b-metric spaces, *J. Math. Comput. Sci.*, 16, 131-139.

[20] Zamfirescu, T., 1972, Fix point theorems in metric spaces, *Arch. Math.*, 23, 292-298.