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Bifurcation Analysis of Impact of Awareness Campaign on Alcoholism with Delay

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Abstract: *In this paper, transmission dynamics of alcoholism with media awareness programs has been studied by developing a mathematical model using delay differential equations. It is assumed that heavy alcohol drinking habit spreads by the direct contact between susceptibles and heavy alcohol consumers. The rate of growth of media awareness campaigns is assumed to be proportional to the number of deaths due to heavy alcohol drinking. We have formulated a compartmental model considering individual's behavioral changes due to the influences of awareness campaigns coverage and converted the unaware susceptible individuals into aware. Time delay factor is considered for the delay of conversion of unaware susceptible individuals into heavy alcoholic by their interaction. The stability analysis of equilibrium points is carried out. The basic reproduction number* R_0 *is computed by applying the next generation matrix approach. Moreover, sensitivity analysis of the parameters involved in* R_0 *is conducted employing normalized forward sensitivity technique. The model analysis revealed that the spread of alcoholism in the community can be controlled by awareness programs but it persists in the community as a drinking-present equilibrium. The drinking-present equilibrium exhibits Hopf bifurcation regarding time delay as the delay parameter. Numerical simulations provide the results of analytical outcomes and the significance of awareness campaigns and delay in mitigating alcoholism in society.*

Keywords: Equilibrium point, Basic reproduction number, Sensitivity analysis, Numerical simulation, Hopf bifurcation.

1 Introduction

Globally, heavy alcohol use creates 2.8% million deaths, which accounts 2.2 % of all deaths among females and 6.8% of all deaths among males [10]. Additionally, it is ranked as the seventh risk factor for both morbidity and mortality. The effects of regular excessive consumption of alcohol on health are anxiety, cancer, depression, epilepsy, heart disease, hypertension, liver cirrhosis. It also has a significant role in increasing social and economic problems, including unemployment, domestic violence, traffic accidents [15, 19]. An estimat 474,000 deaths from cardiovascular diseases were caused by alcohol consumption in 2019 [12]. When an individual has fetal alcohol syndrome, they represent the most extreme case within the range of fetal alcohol spectrum disorders (FASD). This is a permanent condition that results from prenatal alcohol exposure (PAE) and has no cure. It is marked by damage to the central nervous system and significant neurodevelopmental challenges, which may or may not include physical birth defects, along with various associated health issues [18].

According to a population-based survey conducted in Nepal, 53.7% of men reported drinking alcohol recently, and there were correlations between drinking and age, caste, religion, education, occupation, and tobacco usage [20]. It is found that nearly 23.8% of male drinkers screened positive for alcohol use disorders (AUD). It is found that the alcohol use disorders identification test (AUDIT) sores were correlated with age, caste, marital status, occupation, tobacco use, depression, and functional status. The report of the STEPS survey on non-communicable disease risk factors in Nepal, carried out by the Nepal Health Research Council (NHRC) from February to May 2019, revealed that 4.8 million (23.9%) of adults were current drinkers, among them 3.7 million were men and 1.1 million were women [23].

Public health–related behaviors can be influenced through media platforms such as radio, newspapers, television, and online social networks. These channels disseminate up-to-date information on alcoholism and suggest appropriate control measures. Individuals, who are aware of such information, are more likely to adopt preventive practices to reduce their risk of developing alcoholism. Therefore, media awareness campaigns serve as an effective tool for modifying behaviors associated with alcoholism.

Many different mathematical models regarding alcoholism as an epidemic, have been formulated and studied to mimic the dynamics of drinking and control of drinking behavior [1][3] [5] [14]. Chinnadurai et al. (2024) studied alcohol consumption control and its effect on the poor population by categorizing the alcohol consuming people into four compartments [7]. Sher et al. (2024) have considered the concept of conformable fractional order derivative (CFOD) for modelling. They have investigated an alcohol-abuse mathematical model applying the concept of fractional-order piecewise operator for qualitative theory, including existence and uniqueness of solution [21]. Itoro-Obong and Acheneje (2024) have presented a mathematical model for the computation of different parameters of alcoholism epidemics in the context of Nigeria [13]. Their model is based on the assumptions that alcoholism can be mitigated solely through treatment, and it does not take into consideration the impact of the media or the time delay in transmission of alcoholism.

The aim of the research is to study the transmission dynamics of alcoholism, with particular attention to the impact of media awareness campaigns and the time delay involved in the conversion of susceptible individuals into heavy alcoholics. The novelty of the study lies in its comprehensive mathematical modelling of alcoholism, which accounts for the fact that the transition from a non-alcoholic individual to a heavy alcoholic is not instantaneous.

2 Mathematical Model

Our model modifies the previously developed model by Kunwar and Verma (2023) for the spread of alcoholism in society with the effect of a media awareness campaign [14]. In this model, the population as entire is separated into three categories: susceptible group, represented by $S(t)$, that does not drink or drinks occasionally; risk-aware group, represented by $X(t)$, that stays away from drinking; and heavy drinkers, represented by $A(t)$. The cumulative density of awareness measures, denoted by $M(t)$, is considered to be proportional to the number of heavy drinkers. The susceptible and heavy drinking population constitute a conscious class $X(t)$, whose behavior is influenced through ongoing awareness campaigns. Assuming that the cumulative density of media-based campaigns to raise awareness increases at a rate κ_1 proportional to the population of heavy drinkers and falls at a rate m due to factors like inefficiency and psychological barriers. Moreover, the parameter κ_1 is proportional to the alcohol induced death α and implementation rate of awareness programs κ, so that we can write $\kappa_1 = \kappa\alpha$. When occasional drinkers come in contact with heavy drinkers, they will not become heavy alcohol consumers immediately, so we have incorporated the time delay (τ) in the conversion process of alcoholism to justify the phenomenon.

The flow diagram of transmission of alcoholism in different compartments as a communicable disease has been illustrated in the Figure 1.

The following system of delay differential equations governs the dynamics of alcoholism as a communicable disease:

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t-\tau) A(t-\tau) - \mu S(t) M(t) + (1-p)\gamma A(t) + \sigma X(t) - \delta S(t), \\
\frac{dA(t)}{dt} &= \beta S(t-\tau) A(t-\tau) - (\alpha+\delta+\gamma) A(t), \\
\frac{dX(t)}{dt} &= \mu S(t) M(t) + p\gamma A(t) - (\delta+\sigma) X(t), \\
\frac{dM(t)}{dt} &= \kappa\alpha A(t) - mM(t),
\end{aligned}\right\} \qquad (1)$$

with initial conditions : $S(\theta),\ A(\theta),\ X(\theta) \geq 0,\ \theta \in [-\tau, 0]$ and $S(0) > 0, X(0), A(0) > 0$. $N(t)$ represents the entirety of the population at any time t. Using the relation $S(t) = N(t) - A(t) - X(t)$, the

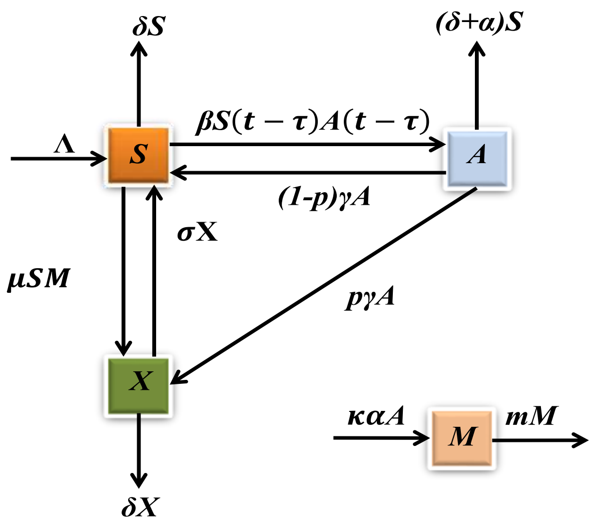


Figure 1: Flow diagram of transmission of alcoholism

system (1) can be re-written as follows :

$$\left.\begin{aligned}\frac{dA(t)}{dt} &= \beta\left[N(t-\tau)-A(t-\tau)-X(t-\tau)\right]A(t)-(\alpha+\delta+\gamma)A(t),\\ \frac{dX(t)}{dt} &= \mu\left[N(t)-A(t)-X(t)\right]M(t)+p\gamma A(t)-(\delta+\sigma)X(t),\\ \frac{dN(t)}{dt} &= \Lambda-\delta N(t)-\alpha A(t),\\ \frac{dM(t)}{dt} &= \kappa\alpha A(t)-mM(t).\end{aligned}\right\} \tag{2}$$

Table 1: Description of the model parameters.

Parameter	Description
Λ	Rate of recruitment of individuals into susceptible class
p	Proportion of recovered heavy drinker that join the aware population
β	Transmission rate of drinking habit to susceptible individuals from the heavy drinkers
μ	Dissemination rate of awareness among unware susceptible
σ	Rate of transfer of aware population to the class of unaware susceptible
δ	Natural death rate
γ	Rate of recovery of heavy drinker by treatment

3 Model Analysis

Solving the system (2) and taking limit as $t \to \infty$, we get the region of attraction. The result is stated as a theorem as follows :

Theorem 1. *The region of attraction of solution of the system* (2) *with the initial conditions is a closed feasible region* Ω*, defined by :*

$$\Omega = \left\{(A(t),\ X(t),\ N(t),\ M(t)) \in \mathbb{R}_+^4 : 0 \le A(t),\ X(t) \le N(t) \le \frac{\Lambda}{\delta},\ 0 \le M \le \frac{\kappa\alpha\Lambda}{m\delta}\right\}$$

Now, we compute equilibrium points as follows:

3.1 Drinking-Free Equilibrium (DFE) Point and Basic Reproduction Number

For DFE point $E_0\left(A_0,\ X_0,\ N_0,\ M_0\right)$, we must have : $\dfrac{dA}{dt}=0;\ \dfrac{dX}{dt}=0;\ \dfrac{dN}{dt}=0;\ \dfrac{dM}{dt}=0$ and $A_0=0$.
Solving the above equations, we get $N_0=\dfrac{\Lambda}{\delta},\ M_0=0$ and $X_0=0$.
Hence, the drinking-free equilibrium point is $E_0\left(0,\ 0,\ \dfrac{\Lambda}{\delta},\ 0\right)$.

The basic reproduction number is the number of new heavy drinkers multiplied by a single heavy drinker. Using next-generation matrix approach [11, 22], the spectral radius of the next-generation matrix is the basic reproduction number, which is found as:

$$R_0=\frac{\beta\Lambda}{\delta\left(\alpha+\delta+\gamma\right)}. \tag{3}$$

3.2 Drinking-Present Equilibrium (DPE) Point

For DPE point $E^*\left(A^*,\ X^*,\ N^*,\ M^*\right)$, $A^*\neq 0$ and $\left.\dfrac{dA}{dt}\right|_{E^*}=0;\ \left.\dfrac{dX}{dt}\right|_{E^*}=0;\ \left.\dfrac{dN}{dt}\right|_{E^*}=0;\ \left.\dfrac{dM}{dt}\right|_{E^*}=0.$
Solving the above equations, we get
$A^*=\dfrac{m\delta d\left(\delta+\sigma\right)\left(R_0-1\right)}{m\delta\beta p\gamma+\mu\kappa\alpha\delta d+m\beta\left(\delta+\sigma\right)\left(\alpha+\delta\right)},\ X^*=\dfrac{\Lambda-\left(\alpha+\delta\right)A^*}{\delta}-\dfrac{\left(\alpha+\delta+\gamma\right)}{\beta},\ N^*=\dfrac{\Lambda-\alpha A^*}{\delta}$, and
$M^*=\dfrac{\kappa\alpha A^*}{m}$, where $d=\alpha+\delta+\gamma$.
It is observed that we will get a unique positive value of A^* only when $R_0>1$. Therefore, when $R_0>1$, a unique DPE point $E^*\left(A^*,X^*,N^*,M^*\right)$ exists.

3.3 Stability Analysis for $\tau=0$

We have the following theorems in this regards :

Theorem 2. *For $\tau=0$, the disease-free equilibrium point $E_0\left(0,\ 0,\ \dfrac{\Lambda}{\delta},\ 0\right)$ is locally asymptotically stable, if $R_0<1$, and unstable, if $R_0>1$.*

Proof. The Jacobian matrix of the system (2) at DFE point E_0 is given by

$$J\left(E_0\right)=\begin{bmatrix}\beta\dfrac{\Lambda}{\delta}-d & 0 & 0 & 0\\ p\gamma & -\left(\delta+\sigma\right) & 0 & \mu\dfrac{\Lambda}{\delta}\\ -\alpha & 0 & -\delta & 0\\ \kappa\alpha & 0 & 0 & -m\end{bmatrix}$$

Solving the characteristic equation for the matrix $J\left(E_0\right)$, we get four eigenvalues $\lambda_1,\lambda_2,\ \lambda_3$ and λ_4 as follows :

$$\lambda_1=-m,\ \lambda_2=-\delta,\ \lambda_3=-\left(\delta+\sigma\right),\ \text{and}\ \ \lambda_4=\frac{\beta\Lambda}{\delta}-d=\left(\alpha+\delta+\gamma\right)\left(R_0-1\right).$$

It is observed that three eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ are real and negative. The fourth eigenvalue λ_4 will also be negative real only, when $R_0<1$. Hence, all four eigenvalues have negative real part if $R_0<1$ and so the drinking-free equilibrium point E_0 is locally asymptotically stable [4].

□

Theorem 3. *For $\tau=0$, the drinking-present equilibrium point E^* is locally asymptotically stable, if the coefficients of the characteristic equation of the system (2) at E^* satisfy the conditions given in (5).*

Proof. The Jacobian matrix at the disease present equilibrium point E^* is given by :

$$J\left(E^*\right) = \begin{bmatrix} \beta\left(N^* - 2A^* - X^*\right) - d & -\beta A^* & \beta A^* & 0 \\ -\mu M^* + p\gamma & -\mu M^* - (\delta+\sigma) & \mu M^* & \mu\left(N^* - A^* - X^*\right) \\ -\alpha & 0 & -\delta & 0 \\ \kappa\alpha & 0 & 0 & -m \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & 0 & x_{33} & 0 \\ x_{41} & 0 & 0 & x_{44} \end{bmatrix}$$

where $x_{11} = \beta\left(N^* - 2A^* - X^*\right) - d,\ x_{12} = -\beta A^*, x_{13} = \beta A^*,\ x_{21} = p\gamma - \mu M^*,\ x_{22} = -\mu M^* - (\delta+\sigma),$ $x_{23} = \mu M^*,\ x_{24} = \mu\left(N^* - A^* - X^*\right),\ x_{31} = -\alpha,\ x_{33} = -\delta, x_{41} = \kappa\alpha,\ x_{44} = -m.$
The characteristic equation of system (2) at E^*, with η as eigenvalue, is given by

$$\eta^4 + A_1\eta^3 + A_2\eta^2 + A_3\eta + A_4 = 0 \tag{4}$$

where

$$\begin{aligned} A_1 &= -\left(x_{11} + x_{22} + x_{33} + x_{44}\right) > 0, \\ A_2 &= x_{22}x_{33} + x_{11}x_{33} + x_{11}x_{22} + x_{11}x_{44} + x_{22}x_{44} + x_{33}x_{44} - \left(x_{12}x_{21} + x_{13}x_{31}\right) > 0, \\ A_3 &= x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{22}x_{31} + x_{12}x_{21}x_{44} + x_{13}x_{31}x_{44} - \left(x_{12}x_{24}x_{41} + x_{11}x_{22}x_{33}\right. \\ &+ \left. x_{22}x_{33}x_{44} + x_{11}x_{33}x_{44} + x_{11}x_{22}x_{44}\right) > 0, \\ A_4 &= x_{12}x_{24}x_{33}x_{41} + x_{11}x_{22}x_{33}x_{44} - \left(x_{12}x_{21}x_{33}x_{44} + x_{12}x_{23}x_{31}x_{44} + x_{13}x_{22}x_{31}x_{44}\right) > 0. \end{aligned}$$

As $x_{ii} < 0 \quad \forall\ i = 1,2,3,4,\ x_{12} < 0,\ x_{31} < 0,$ and $x_{13} > 0,\ x_{21} > 0,\ x_{23} > 0,\ x_{24} > 0,\ x_{41} > 0,$ the coefficients $A_i \quad \forall\ i = 1,2,3,4,$ all are positive. Furthermore, if the following conditions are satisfied, the eigenvalues have negative real components in accordance with the Routh-Hurwitz criterion :

$$A_1A_2 - A_3 > 0, \quad \text{and} \quad A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0 \tag{5}$$

Hence, drinking-present equilibrium E^* is locally asymptotically stable when the condition (5) holds. □

3.4 Stability Analysis for $\tau > 0$

The impact of delay (τ) on the stability of the drinking-present equilibrium point E^* is described in the following theorem :

Theorem 4. *If $R_0 > 1$ and the coefficients $a_i;\ i = 1,2,3$ of cubic equation (9) satisfy the Routh-Hurwitz criterion (10), then the drinking-present equilibrium point E^* is locally asymptotically stable for all $\tau > 0$.*

Proof. For the linearization of the system (2) about the DPE point E^*, we substitute
$A = A^* + x,\ X = X^* + y,\ N = N^* + n,$ and $M = M^* + z.$ Then, linearized system gets the form :

$$\frac{du\left(t\right)}{dt} = M_1\ u\left(t\right) + M_2\ u\left(t - \tau\right) \tag{6}$$

where

$$\begin{aligned} u\left(t\right) &= \left[x\left(t\right),\ y\left(t\right),\ n\left(t\right),\ z\left(t\right)\right]^T, \\ M_1 &= \begin{bmatrix} -d & 0 & 0 & 0 \\ p\gamma & -(\delta+\sigma) & 0 & 0 \\ -\alpha & 0 & -\delta & 0 \\ \kappa\alpha & 0 & 0 & -m \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix}, \\ M_2 &= \begin{bmatrix} -\beta A^* & -\beta A^* & \beta A^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where
$b_{11} = -d < 0,\ b_{22} = -(\delta+\sigma) < 0,\ b_{33} = -\delta < 0,\ b_{44} = -m < 0,\ b_{21} = p\gamma > 0,\ b_{31} = -\alpha < 0,\ b_{41} = \kappa\alpha > 0,\ b_{44} = -m < 0,\ c_{11} = -\beta A^*,\ c_{12} = -\beta A^*,\ c_{13} = \beta A^*.$
The characteristic equation $\left|M_1 + M_2 e^{-\eta\tau} - \eta I\right| = 0$ of the system (6), with η as its eigenvalue and I identity matrix of order four, is as follows:

$$\begin{vmatrix} b_{11} + c_{11}e^{-\eta\tau} - \eta & c_{12}\ e^{-\eta\tau} & c_{13}\ e^{-\eta\tau} & 0 \\ b_{21} & b_{22} - \eta & 0 & 0 \\ b_{31} & 0 & b_{33} - \eta & 0 \\ b_{41} & 0 & 0 & b_{44} - \eta \end{vmatrix} = 0$$

Expanding, we get one of the eigenvalues as $\eta_1 = b_{44} < 0$, and the remaining eigenvalues are the roots of the equation:

$$\eta^3 + c_1\eta^2 + c_2\eta + c_3 + e^{-\eta\tau}\left[d_1\eta^2 + d_2\eta + d_3\right] = 0 \tag{7}$$

where

$$\begin{aligned} c_1 &= (d + 2\delta + \sigma) > 0,\ c_2 = \left(2d\delta + d\sigma + \delta^2 + \delta\sigma\right) > 0,\ c_3 = \{d\delta\,(\delta+\sigma)\} > 0,\ d_1 = \beta A^* > 0, \\ d_2 &= \{(p\gamma + \alpha)\,\beta - \beta\sigma\}\,A^* > 0,\ d_3 = \{\alpha\beta\gamma p A^* + \alpha\beta\,(\delta+\sigma)\} > 0. \end{aligned}$$

The required conditions for the stability of E^* and the occurrence of Hopf bifurcation will now be derived. When $\tau > 0$, the cubic equation (7) representing the characteristic equation, do not have non-negative real roots. Its roots must be complex and derived from a pair of complex conjugate roots that cross the imaginary axis if they contain non-negative real component. As a result, for some $\tau > 0$, the equation (7) must have two purely imaginary roots. Without loss of generality, we assume that $\eta = i\omega \quad (\omega > 0)$ is a root of the equation (7). So, we have

$$-i\left(\omega^3 - c_2\omega\right) + \left(c_3 - c_1\omega^2\right) + e^{-i\omega\tau}\left[(d_3 - d_1\omega^2) + i(d_2\omega)\right] = 0$$

Writing $x = \omega\tau$ and equating real and imaginary parts, we get

$$\left.\begin{aligned} (d_3 - d_1\omega^2)cosx + d_2\omega\ sinx &= c_1\omega^2 - c_3 \\ d_2\omega\ cosx - (d_3 - d_1\omega^2)\ sinx &= \omega^3 - c_2\omega \end{aligned}\right\} \tag{8}$$

To eliminate the trigonometrical functions, squaring and adding the two equations of (8) and denoting $\omega^2 = \phi$, we get

$$f(\phi) = \phi^3 + a_1\phi^2 + a_2\phi + a_3 = 0 \tag{9}$$

where $a_1 = (c_1^2 - 2c_2 - d_1^2),\ a_2 = (c_2^2 - 2c_1c_3 + 2d_1d_3),\ a_3 = (c_3^2 - d_3^2).$
On the basis of Routh-Hurwitz criterion, all of the roots of the characteristic equation (9) have a negative real component, if the following conditions are achieved :

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ \text{and}\ a_1a_2 - a_3 > 0 \tag{10}$$

Hence, if $a_i;\ i = 1,2,3$ of equation (9) satisfy the above criterion, we do not get any positive value of ϕ. Thus, the drinking-present equilibrium point E^* is stable, if the conditions (10) hold. □

4 Bifurcation Analysis

For basic reproduction number less than 1 (by varying β), the equilibrium values of A^* are essentially zero indicating the drinking-free state with no heavy alcoholic population (Figure 2). Furthermore, for $R_0 > 1$, the graph shows $A^* > 0$ that grows continuously from zero as R_0 crosses 1 forming the endemic branch. As the endemic branch emerges continuously from zero at $R_0 = 1$, the bifurcation is a forward (supercritical) bifurcation. Hence, the endemic state is found to gain smoothly as R_0 increases past unity.

Hopf bifurcation occurs when the delay exceeds the critical value that results from the presence of purely

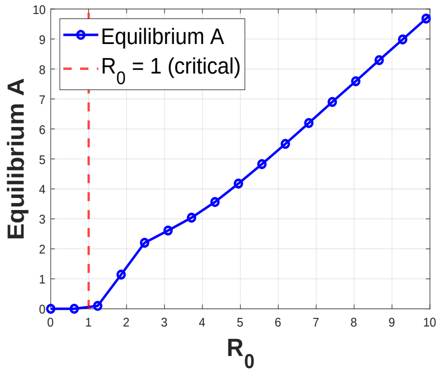


Figure 2: Transcritical forward bifurcation: steady value of heavy alcoholic population A^* is plotted against R_0 taking parameters from table 2 except β, the value of β varies in the interval (0, 0.0005).

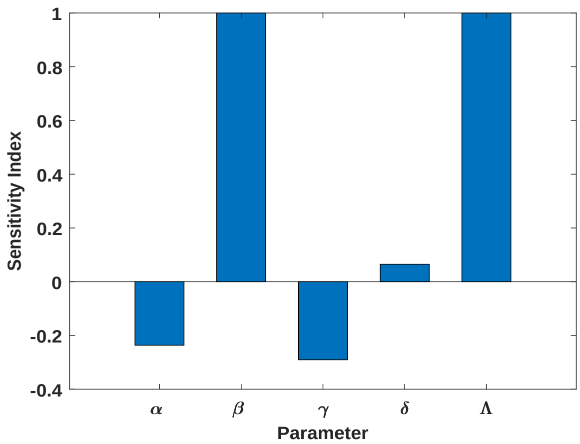


Figure 3: Sensitivity indices of the parameters involved in R_0, the values of parameters are taken as indicated in Table 2

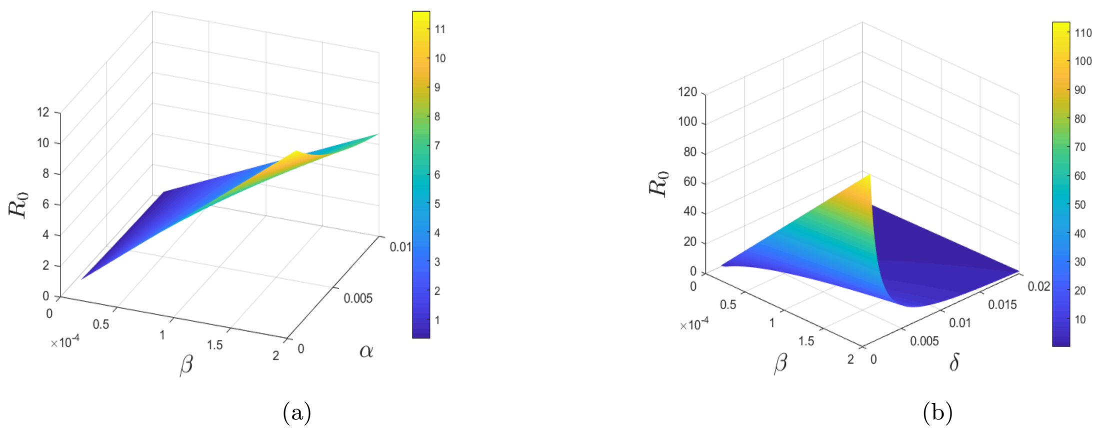
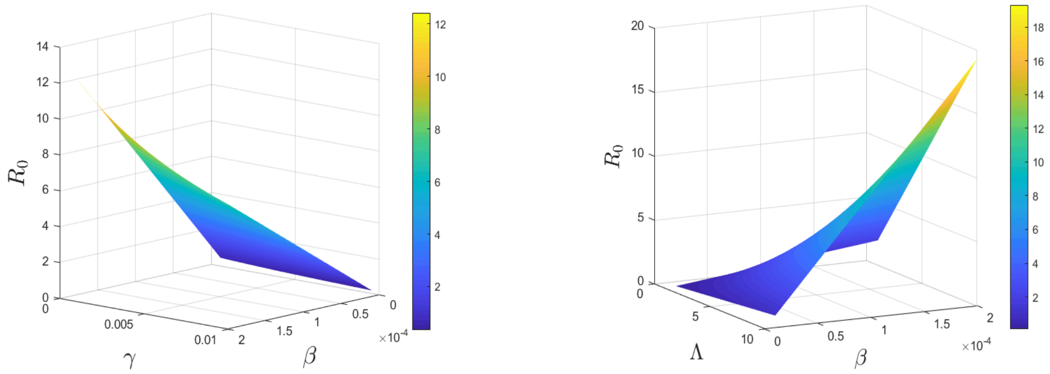


Figure 4: Surface plot of R_0: (a) in $\beta - \alpha$, (b) in $\beta - \delta$ planes using the values of parameters as indicated in Table 2 and $\tau = 10$ days.

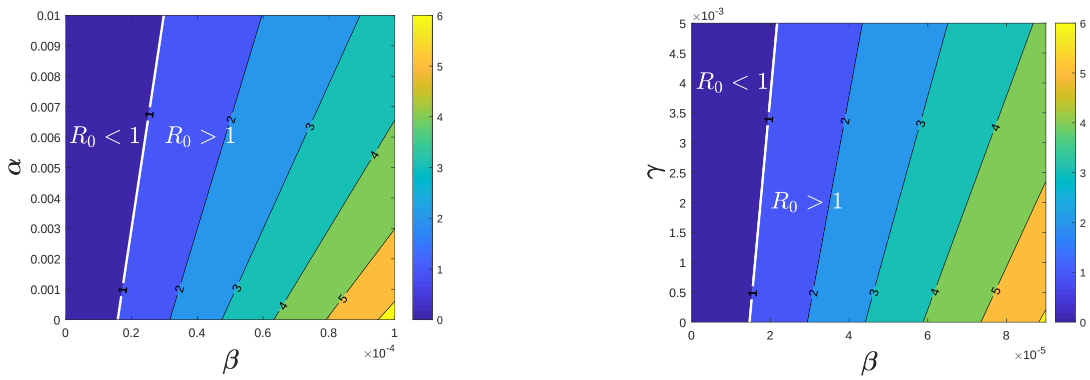


(a) (b)

Figure 5: Surface plot of R_0: (a) $\beta - \gamma$, and (b) $\beta - \Lambda$ planes using the values of parameters as indicated in Table 2 and $\tau = 10$ days.

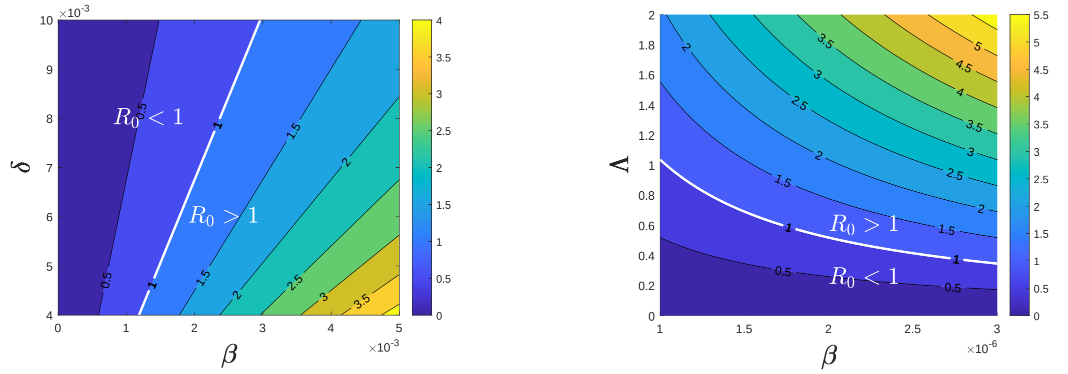
Table 2: The values of parameters used for simulations.

Parameter & Value	Source	Parameter & Value	Source
$\Lambda = 5$	[16]	$m = 0.02$	[2]
$\beta = 0.00008$	(Assumed and vary)	$\mu = 0.008$	Assumed
$\sigma = 0.001$	[16]	$\gamma = 0.0043$	Assumed
$\delta = 0.007$	[17]	$p = 0.15$	[24]
$\alpha = 0.0035$	[9]	$\kappa = 2.5$	Assumed



(a) (b)

Figure 6: Contour plot of R_0: (a) in $\beta - \alpha$, (b) in $\beta - \gamma$ planes using the values of parameters as indicated in Table 2 and $\tau = 10$ days.



(a) (b)

Figure 7: Contour plot of R_0: (a) $\beta - \delta$, and (b) $\beta - \Lambda$ planes using the values of parameters as indicated in Table 2 and $\tau = 10$ days.

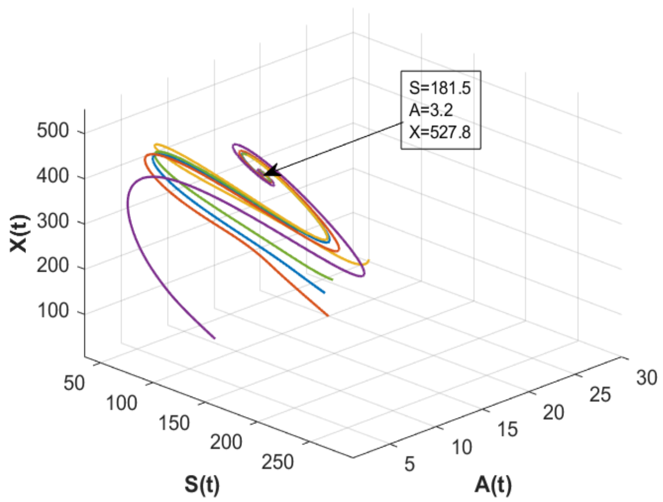
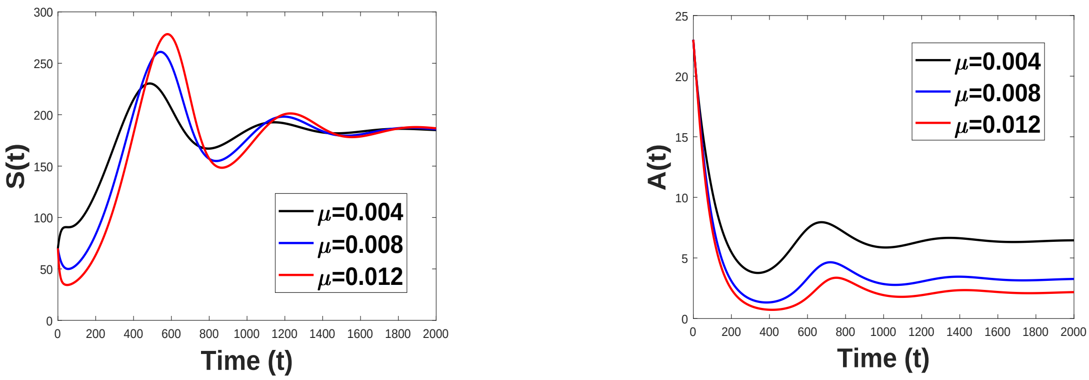
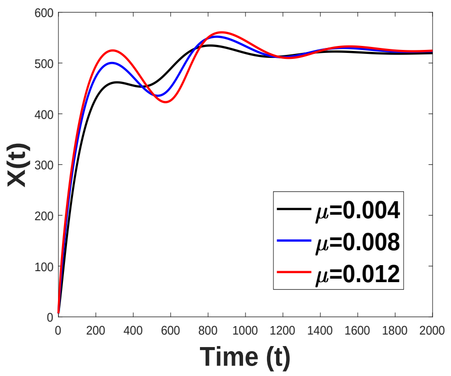


Figure 8: The stability of the drinking-present equilibrium with $\tau = 10$ and $R_0 = 3.9 > 1$.

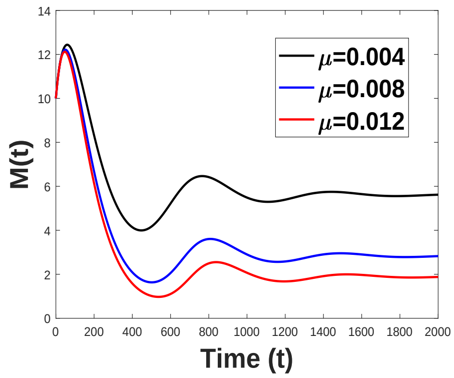


(a) (b)

Figure 9: Trajectories of (a) susceptible (b) heavy drinker for different values of dissemination rate of awareness programs (μ)among susceptible population.



(a)



(b)

Figure 10: Trajectories of (a) aware populations, and (b) cumulative density of awareness program for different values of dissemination rate of awareness programs (μ) among susceptible population.

imaginary roots in the characteristic equation. If the two assumptions H_1 and H_2 are true, then the model system exhibits a local Hopf bifurcation at the drinking-present equilibrium point for the critical value τ_0.
H_1**:** Only one pair of imaginary roots, $\pm i\omega$, and all other negative real parts of roots appear in the characteristic equation at E^*,
H_2**:** $\left[Re(\frac{d\eta}{d\tau})^{-1}\right]\Big|_{\eta=i\omega_0} \neq 0$.

To establish the assumption H_1, it is necessary that the provided equation accommodate a minimum of one positive root. By applying Descartes' Rule of Signs, we may confirm the existence of such a root. Suppose the equation possesses at least one positive root. Consequently, the equation will yield a purely imaginary root at the critical value τ_0. From this, we determine the corresponding value as :

$$\tau_0 = \frac{1}{\omega_0}\cos^{-1}(\chi(\omega_0)). \text{ where, } \chi(\omega_0) = \frac{(c_1\omega^2 - c_3)(d_3 - d_1\omega^2) + (\omega^3 - c_2\omega)(d_2\omega)}{(d_3 - d_1\omega^2)^2 + (d_2\omega)^2}.$$

Further, we differentiate the characteristic equation with respect to τ and find reciprocal of the real part as follows :

$$\left(\frac{d\eta}{d\tau}\right)^{-1}\Bigg|_{\eta=i\omega_0,\ \tau=\tau_0} = \frac{-3\omega_0^2 + 2ic_1\omega_0 + c_2 + e^{-i\omega_0\tau_0}\Big(2id_1\omega_0 + d_2 + \tau_0 d_1\omega_0^2 - i\tau_0 d_2\omega_0 - \tau_0 d_3\Big)}{i\omega_0 e^{-i\omega_0\tau_0}\big(-d_1\omega_0^2 + id_2\omega_0 + d_3\big)}.$$

Therefore, we have

$$\Re\left[\left(\frac{d\eta}{d\tau}\right)^{-1}\Bigg|_{\eta=i\omega_0,\ \tau=\tau_0}\right] = \frac{M'}{\omega_0\Big[\big(-d_1\omega_0^2 + d_3\big)^2 + (d_2\omega_0)^2\Big]},$$

where

$$\begin{aligned} M' = &-d_2\omega_0\big(-3\omega_0^2 + c_2\big)\cos(\omega_0\tau_0) + d_2\omega_0\big(2c_1\omega_0\big)\sin(\omega_0\tau_0) \\ &+ \Big[\big(-3\omega_0^2 + c_2\big)\sin(\omega_0\tau_0) + \big(2c_1\omega_0\big)\cos(\omega_0\tau_0) + 2d_1\omega_0 - \tau_0 d_2\omega_0\Big]\big(-d_1\omega_0^2 + d_3\big) \\ &- d_2^2\omega_0 - \tau_0 d_1 d_2\,\omega_0^3 + \tau_0 d_2 d_3\,\omega_0. \end{aligned}$$

Hence, we have $\Re\left[\left(\frac{d\eta}{d\tau}\right)^{-1}\Big|_{\eta=i\omega_0,\ \tau=\tau_0}\right] > 0$, if $M' > 0$.
Therefore, the transversality condition is satisfied, and the system undergoes Hopf bifurcation at $\tau = \tau_0$.

5 Sensitivity Analysis

We use the approach in Chitnis et al. (2008) to calculate the normalized forward sensitivity indices of R_0 with respect to parameters involved [8].

The algebraic representation of the sensitivity index R_0 with respect to the parameters $\beta, \delta, \alpha, \gamma$ are as follows:

$$S_\beta^{R_0} = 1,\ S_\alpha^{R_0} = \frac{-\alpha}{\alpha+\delta+\gamma},\ S_\gamma^{R_0} = \frac{-\gamma}{\alpha+\delta+\gamma},\ S_\delta^{R_0} = \frac{-(\alpha+2\delta+\gamma)}{\alpha+\delta+\gamma},\ S_\Lambda^{R_0} = 1.$$

From Figure 4a, it is clear that R_0 rises sharply as β and α increase. This is because of the fact that higher mortality corresponds to a longer alcoholism period allowing more secondary transmission before death. When both β and α are low, the value of R_0 becomes less than unity, so the alcoholism can not persist. Otherwise, when β is high, even a small increase in α drives R_0 well above unity making alcoholism outbreak.

When β is high, even a modest increase in the value of γ may not immediately bring R_0 below unity. Moreover, when the values of β is small, moderate treatment coverage and an increase in the value of γ can rapidly bring R_0 less than unity (Figure 4b). For a larger value of δ, there will be heavier natural mortality resulting heavy alcoholism leave the alcoholic class faster and thus reduces the value of R_0 (Figure 5a). Figure 5b indicates that if Λ is small, even moderately high value of β may not yield a large outbreak because there are not many susceptible to recruit. Also, if Λ is large, even there is a moderate value of β can sustain $R_0 > 1$, resulting the control of alcoholism harder. This indicates that β and Λ are important parameters to address the alcoholic addiction problem. The results are also justified by the patterns in contour plots in Figures 6 and 7 .

6 Numerical Simulations

Numerical simulations are conducted to examine the dynamics of the system and to verify the analytical results. For numerical simulations, we select values of the parameters listed in the Table 2. Referring to the set of parameter values, the basic reproduction number (R_0) is found to be 3.9.

From Figures 9 and 10, it is apparent that awareness campaigns are effective means for managing excessive alcoholism. While the other parameters remain unchanged, we observe that when the value of the dissemination rate μ of awareness campaign among susceptible populations rises, the aware population $X(t)$ grows while heavy drinkers $A(t)$ declines. Therefore, the implementation of awareness campaigns should be undertaken to control alcohol-related issues.

Furthermore, the corresponding expression can be employed to determine the critical value of time delay $\tau_0 = 76$ based on the parameter values listed previously. The Figure 11 shows trajectories of solutions for time delay τ less than the critical value τ_0. It is observed that for lower value of τ, the model populations oscillate first and then before convergent to the equilibrium E^*. This indicates that the number of heavy drinkers population will fluctuate between high and low numbers so it may be challenging to accurately forecast the severity of alcoholic epidemic initially. The system takes more time to become stable. However, when the time delay τ exceeds the threshold value $\tau_0 = 76$, all the populations exhibit periodic oscillations, indicating that solutions bifurcate into periodic solutions as shown in the Figure 12. The Figures 11 and 12 demonstrate that when $\tau = 70 < \tau_0$ the drinking-present equilibrium E^* of the system (2) is locally asymptotically stable; and Hopf bifurcation occurs when $\tau = 82 > \tau_0$. These figures demonstrate that when the delay parameter τ exceeds the critical threshold τ_0, the equilibrium point loses stability and a limit cycle emerges. The phase plots presented in Figures 13, 14,15, 16 best illustrate the resulting bifurcation patterns.

7 Conclusions

In this study, we have extended the drinking-transmission model proposed by Kunwar and Verma [14] by introducing a time delay in the transition from non-drinkers to heavy drinkers. The resulting delayed

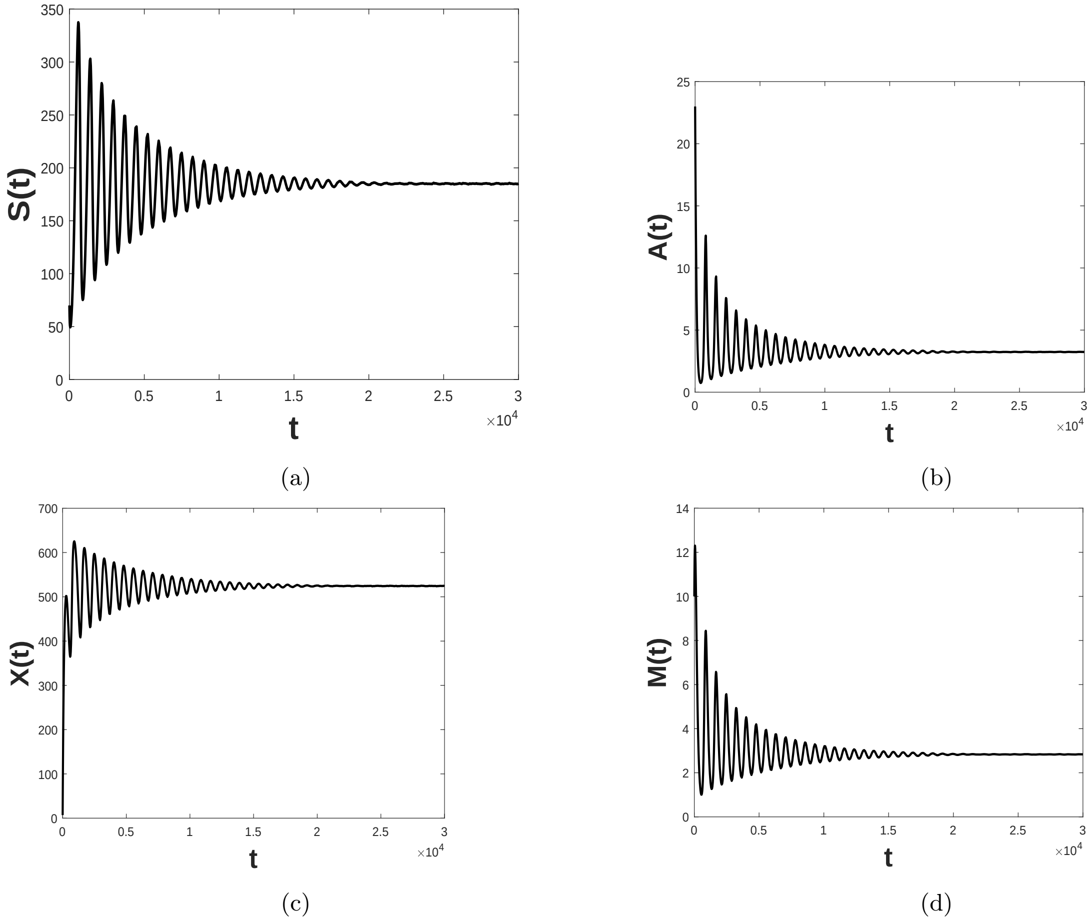


Figure 11: The stability of alcohol-present equilibrium E^* with $\tau = 71 < \tau_0$ and $R_0 = 3.9 > 1$.

model undergoes a rigorous analytical investigation to explore its key dynamical properties, including the existence and stability of equilibrium points, the occurrence of bifurcation, and the sensitivity of system parameters. Our findings suggest that controlling alcohol addiction is feasible when R_0 is below 1, resulting in a stable drinking-free equilibrium. When R_0 exceeds 1, a stable alcohol-present equilibrium emerges, indicating the persistence of heavy drinking behaviors. The existence of a Hopf bifurcation is obtained for the time delay system. The increment of delays beyond a critical threshold value (τ_0) can induce Hopf bifurcation, leading to oscillatory behavior in transmission dynamics of alcoholic patterns. The phase diagrams between different state variables illustrate how time delays interact to influence the stability and dynamics of the alcoholic population. The sensitivity analysis reveals parameters that can substantially decrease R_0, thereby mitigating the spread of heavy alcoholic behaviors. These results underscore the significance of implementing timely and sustained awareness programs to prevent the persistence and resurgence of heavy drinking behaviors. By emphasizing the importance of awareness programs and understanding the role of time delay, this study provides valuable insights for policymakers and healthcare professionals seeking to develop effective strategies for controlling alcohol addiction. The results suggest that prompt and effective media awareness programs can significantly reduce the prevalence of alcoholism. We recommend that the government and relevant health organizations promptly disseminate information on the impacts of alcohol consumption and the associated preventive measures through comprehensive media campaigns. Alcoholism among young people has become a significant societal concern, and further research addressing the various issues arising from it is essential for future mitigation efforts.

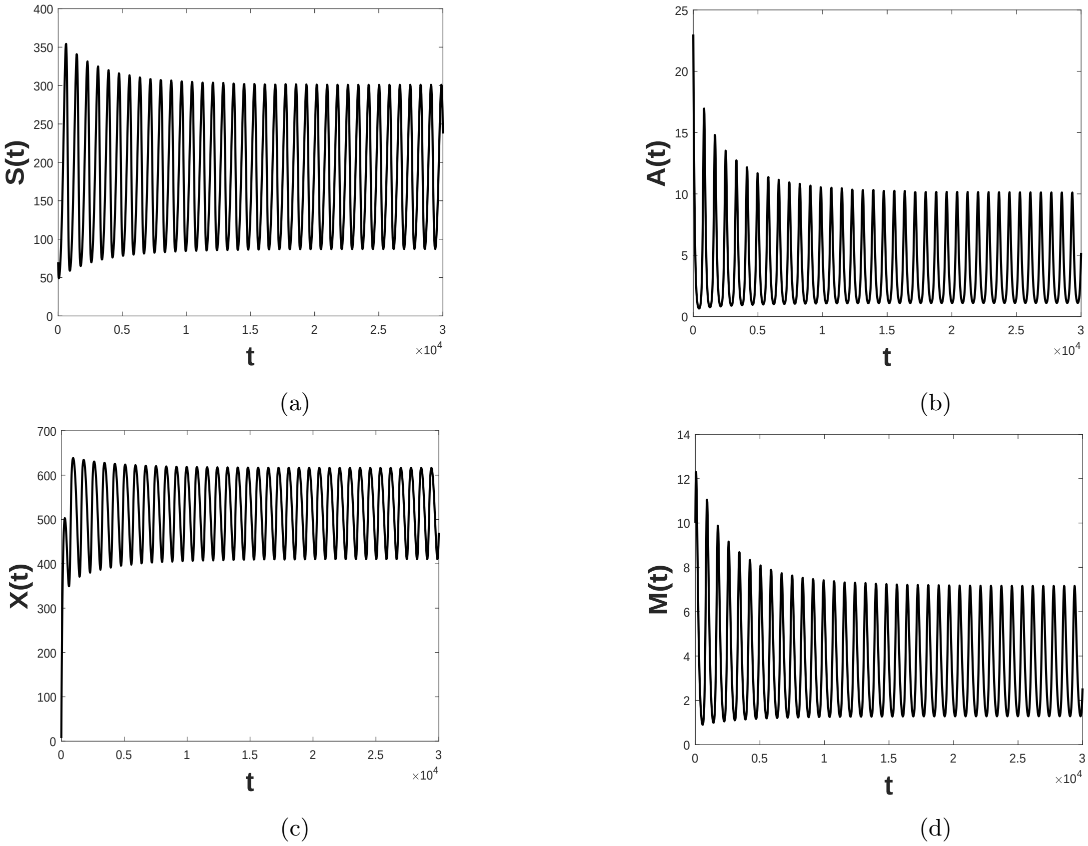


Figure 12: Hopf bifurcation occur with $\tau = 82 > \tau_0$ and $R_0 = 3.9 > 1$.

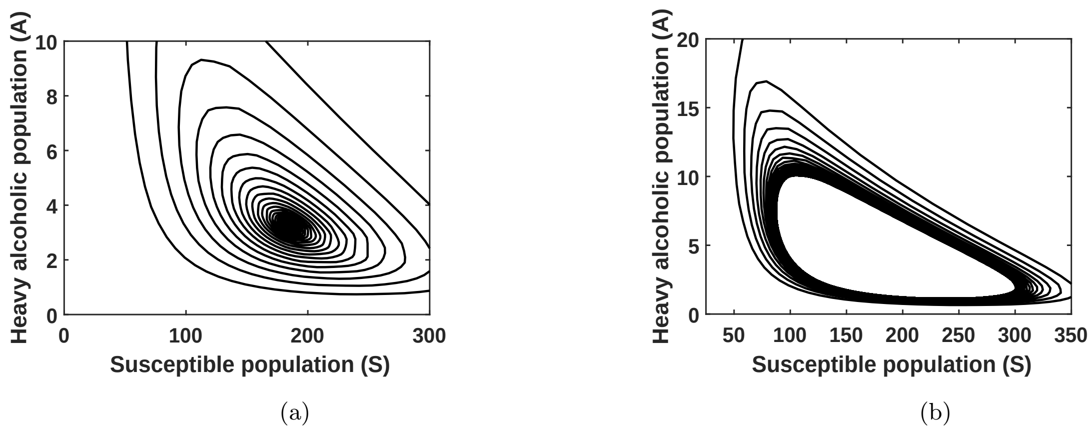
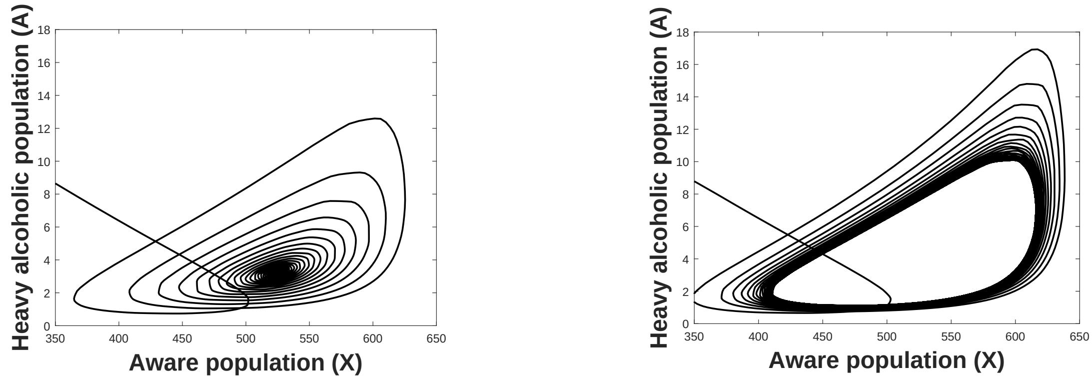
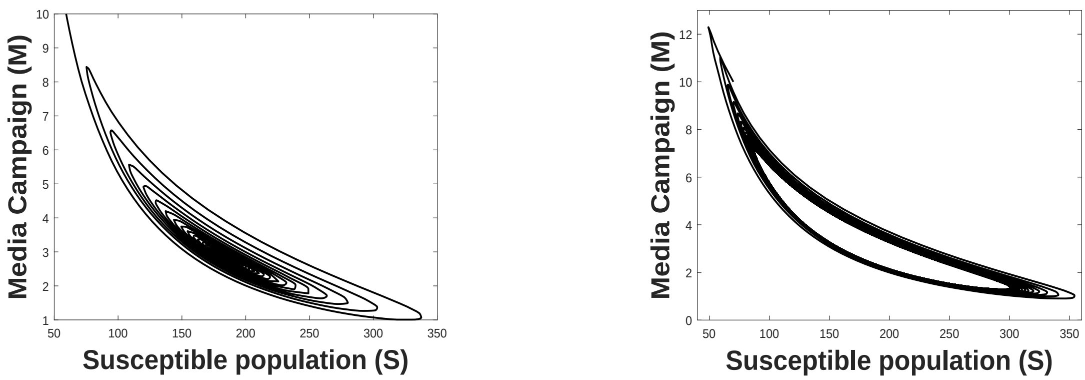


Figure 13: The phase plot of system (2) in $S(t) - A(t)$ plane with (a) $\tau = 70 < \tau_0$, (b) $\tau = 82 > \tau_0$ and $R_0 = 3.9 > 1$.



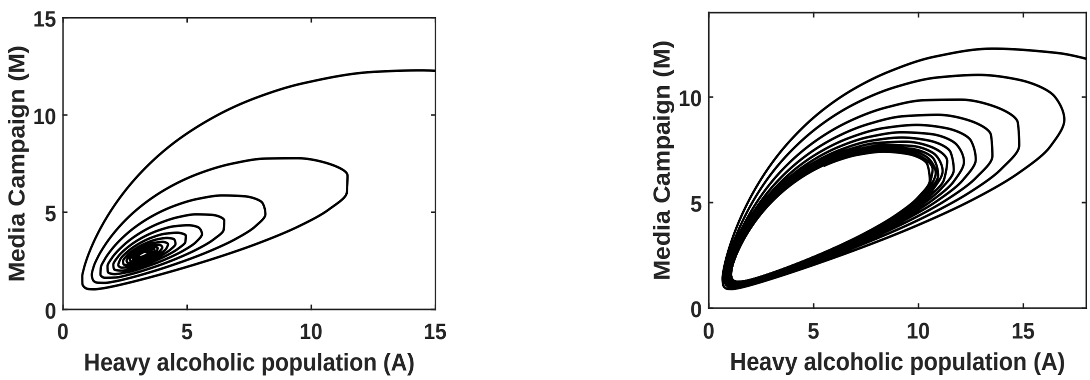
(a) (b)

Figure 14: The phase plot of system (2) in $X(t) - A(t)$ plane with (a) $\tau = 70 < \tau_0$, (b) $\tau = 82 > \tau_0$ and $R_0 = 3.9 > 1$.



(a) (b)

Figure 15: The phase plot of system (2) in $S(t) - M(t)$ plane with (a) $\tau = 70 < \tau_0$, (b) $\tau = 82 > \tau_0$ and $R_0 = 3.9 > 1$.



(a) (b)

Figure 16: The phase plot of system (2) in $A(t) - M(t)$ plane with (a) $\tau = 70 < \tau_0$, (b) $\tau = 82 > \tau_0$ and $R_0 = 3.9 > 1$.

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