Journal of Nepal Mathematical Society (JNMS)
ISSN: 2616-0153 (Print), 2616-0161 (Online)
Vol. 8, Issue 2, 2025 (Dec.): 48-57
DOI: https://doi.org/10.3126/jnms.v8i2.87708

Research Article
Received Date: Oct. 05, 2025
Accepted Date: Dec. 04, 2025
Published Date: Dec. 29, 2025

Accuracy of Numerical Methods in the Absence of Closed-Form Solutions for First Order ODEs

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Abstract: *Ordinary Differential Equations (ODEs) are essential in modeling various real-world phenomena. Finding the analytical solutions to all ODEs is challenging task. In addition, there are very few closed-form solutions. There is a gap in the analysis of ODEs whose solutions exist analytically but cannot be expressed in terms of elementary functions. Majority of the current studies exclusively compare numerical solutions of ODEs with elementary closed-form solutions. This article deals on solving initial value problems having closed form and lacking closed form solutions and compare their accuracy by using Euler method, Improved Euler and RK−4 method. With the idea from closed form solution, this article focuses on the solution of the differential equations to case when the closed form solution is not available.*

Keywords: Closed form, Error function, Euler method, Improved Euler method, RK−4, Error analysis.

1 Introduction

Differential equations can describe all systems undergoing change. They are commonly found in science, engineering, economics, social science, biology, business, health care, and many other fields. Researchers and mathematicians have studied the nature of differential equations and many complicated systems that can be described quite precisely with mathematical expression. Analytical methods as the name suggest involve 'analysis' of the differential equation to get the solutions [13]. Some of the analytical methods to solve ODEs are separation of variables, Integrating factor method, finding the complementary solution and particular integral, special substitution method, method of undetermined coefficients etc. [2]. Analytical solution obtained by using analytical methods are in different forms. One of them is closed form solution. A closed form solution of an ordinary differential equation (ODE) is an exact analytical solution expressed as a finite combination of elementary functions (polynomials, exponential, trigonometric, logarithms, etc). There are some differential equations which do not have the closed form solution or sometimes it is extremely difficult to find. Numerical methods are useful to approximate the solution when it is difficult to obtain closed form solutions. Some of the numerical methods to solve ODEs are Euler method, Improved Euler method, RK−4 method.

The fourth-order Runge-Kutta method was used to solve initial value problems for ordinary differential equations in the work of [9]. The obtained results are close to the exact solutions, especially when the step size is very small and the method was shown to be effective. In this work, ODEs with closed form solution only was considered. The author in [7] compared Euler method, Modified Euler method (MEM), and the Fourth-Order Runge–Kutta method (RK−4) for solving first-order ODEs with initial value problems, which have closed-form solutions. This work established the higher accuracy and faster convergence of RK−4 compared to Euler method and Modified Euler method. The author in [8] used Euler method to solve initial value problems for ordinary differential equations and demonstrated that reducing the step size enhances the accuracy. He also used RK−4 method and showed that the method performed well in comparison with the other methods. The authors in [13] compared the three numerical methods for solving initial value problems(IVPs). They used MATLAB to calculate numerical solutions and compared the exact and numerical solutions. The authors in [16] studied the algorithmic steps, computational complexity and drawbacks of Euler method. They found that the method relies on the step size. Also, they compared it with the higher order methods such as RK−4 method, Adams-Bashforth, etc.

Although several studies have compared Euler, Modified Euler, and RK−4 methods for the first order ODEs, nearly all of them restrict their analysis to the equations with known closed-form solutions. For example, Imran et al.[7] and Kamruzzan and Nath [13] evaluated accuracy by comparing numerical values directly to exact expressions. These works illustrated the convergence behavior of the methods but do not examine the situations where the analytical solution is unavailable or expressed only through special functions.

In this paper, we take a larger approach, looking at both sorts of ODEs: those with closed-form solutions and those without. For ODEs that do not have closed-form solutions interms of elementary functions, we derive analytical solutions purely from the error function in order to compare them accurately with the solution obtained from numerical approaches. We use Euler, Improved Euler, and the RK-4 methods to approximate solutions and compares the results of different numerical methods with analytical solutions obtained using the error function and assess their accuracy and reliability. We also present the convergence, stability, error analysis and sensitivity of the numerical methods that are used.

2 Numerical Methods

Analytical techniques work well for particular types of differential equations, especially linear, separable, exact, homogeneous, and Bernoulli equations. When differential equations do not fit in these forms, or when they are too complex to solve analytically, for example beyond these criteria such as non linear equation or those with variable coefficient, often cannot be solved analytically [10]. Most of the time even for the linear ODEs too, we can not find the closed form solutions. We need numerical methods to solve such type of problem which give approximate solutions. We use three numerical methods which are Euler, Improved Euler and Runge Kutta methods.

Euler method approximates the solution of an ODE by breaking down the interval into small steps and calculating the function values at each step. In general, we have $y_{n+1} = y_n + hf(x_n, y_n)$[7, 11]. Improved Euler's method, also known as Heun method or Runge-Kutta 2nd order method [1, 14] which uses an iterative approach similar to the Euler method, but instead of using just the initial slope, it refines the approximation by averaging two slopes. The iterative formula for the Improved Euler method is given by $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)]$ where, $x_{n+1} - x_n = h$[11]. The Runge-Kutta methods are characterized by their order in the sense that they agree with Taylor's series solution up to terms of h^r , where r is the order of method [2, 15]. The general formula for Runge-Kutta approximation is given by $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where, $k_1 = hf(x_n, y_n) = hf,\ \ k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$, $k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, and $k_4 = hf(x_n + h, y_n + k_3)$ [12, 15].

3 Numerical Computations

In this section, we consider three ordinary differential equations (ODEs) for our study. The first ODE has a closed-form solution. The other two ODEs do not have closed-form solutions, which makes finding their exact answers more challenging. To overcome this, we use special functions that allow us to represent the exact solutions in the form which is suitable for analysis. This approach ensures that our comparison between exact and numerical solutions remain fair and consistent. We then compute numerical approximations for all the three ODEs using three different methods and calculate the errors by comparing these approximations with the exact solutions. Finally, we present our results using table and graphs, which visually demonstrate the accuracy and performance of the numerical methods we use. Here, we use the uniform step sizes $h = 0.1$ and $h = 0.0125$. These values are chosen to illustrate both coarse- step and fine-step behavior. All computations were performed in MATLAB R2023a using double precision arithmetic. The error is calculated by using the formula $E_n = |y(x_n) - y_n|$ for each of the methods used.

Problem 1: We consider the initial value problem $y' = 3x^2 + y, y(0) = 1$ on the interval $0 \leq x \leq 1$ [7].The exact solution of the given problem is $y(x) = -3(x^2 + 2x + 2) + 7e^x$.

Problem 2: We consider the initial value problem $y' = x^2 - xy, y(0) = 1$ on the interval $0 \leq x \leq 1$. The exact solution of the given problem is $y(x) = x - \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} erfi(\frac{x}{\sqrt{2}}) + e^{-\frac{x^2}{2}}$.

Table 1: Numerical approximations and maximum errors for step size $h = 0.1000$

x_n	**Euler's Method**		**Improved Euler Method**		**Runge-Kutta Method**		**Exact Solution**
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	1.100000000000000	6.1964265e-03	1.106500000000000	3.0357347e-04	1.106196458333333	3.1803799e-08	1.106196426529534
0.2	1.213000000000000	1.6819307e-02	1.230332500000000	5.1319288e-04	1.229819311686632	4.5654438e-09	1.229819307121188
0.3	1.346300000000000	3.2711653e-02	1.379617412500000	6.0575947e-04	1.379011558546141	9.4485880e-08	1.379011653032022
0.4	1.507930000000000	5.4842883e-02	1.563327240812500	5.5435732e-04	1.562772603334738	2.8015416e-07	1.562772883488893
0.5	1.706723000000000	8.4325895e-02	1.791376601097812	3.2770620e-04	1.791048325337955	5.6956294e-07	1.791048894900896
0.6	1.952395300000000	1.2243630e-01	2.074721144213083	1.1045852e-04	2.074830620254019	9.8247954e-07	2.074831602733562
0.7	2.255634830000000	1.7063412e-01	2.425466864355456	8.0208794e-04	2.426267410611651	1.5416817e-06	2.426268952293338
0.8	2.628198313000000	2.3058819e-01	2.856990885112779	1.7956143e-03	2.858784226075187	2.2733721e-06	2.858786499447275
0.9	3.083018144300000	3.0420363e-01	3.384074928049621	3.1468500e-03	3.387218570451703	3.2076469e-06	3.387221778098645
1.0	3.634319958730000	3.9365284e-01	4.023052795494832	4.9200037e-03	4.027968420188250	4.3790251e-06	4.027972799213320

Table 2: Numerical approximations and maximum errors for step size $h = 0.0125$

x_n	**Euler's Method**		**Improved Euler Method**		**Runge-Kutta Method**		**Exact Solution**
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	1.105320937064189	8.7548947e-04	1.106200963805844	4.5372763e-06	1.106196426533248	3.7136960e-12	1.106196426529534
0.2	1.227472691959278	2.3466152e-03	1.229826759808037	7.4526868e-06	1.229819307112627	8.5611518e-12	1.229819307121188
0.3	1.374483017290997	4.5286357e-03	1.379020007936936	8.3549049e-06	1.379011652991788	4.0233816e-11	1.379011653032022
0.4	1.555218454236098	7.5544293e-03	1.562779671720286	6.7882314e-06	1.562772883393647	9.5246921e-11	1.562772883488893
0.5	1.779471974289856	1.1576921e-02	1.791051118283938	2.2233830e-06	1.791048894722741	1.7815549e-10	1.791048894900896
0.6	2.058059778358067	1.6771824e-02	2.074825649788754	5.9529448e-06	2.074831602439338	2.9422420e-10	2.074831602733562
0.7	2.402928210008769	2.3340742e-02	2.426250502396311	1.8449897e-05	2.426268951843826	4.4951198e-10	2.426268952293338
0.8	2.827271839672799	3.1514660e-02	2.858750413613196	3.6085834e-05	2.858786498796297	6.5097749e-10	2.858786499447275
0.9	3.345663887002076	4.1557891e-02	3.387161974637892	5.9803461e-05	3.387221777192030	9.0661567e-10	3.387221778098645
1.0	3.974200270551608	5.3772529e-02	4.027882112284272	9.0686929e-05	4.027972797987712	1.2256072e-09	4.027972799213320

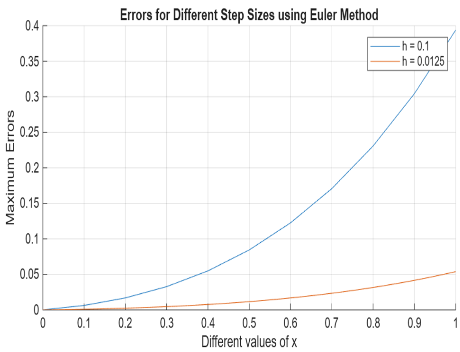

Figure 1: Error for different step sizes using Euler method.

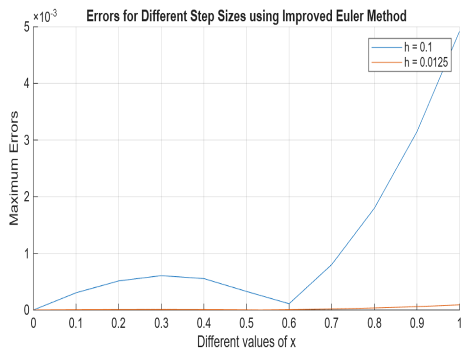

Figure 2: Error for different step sizes using Improved Euler method.

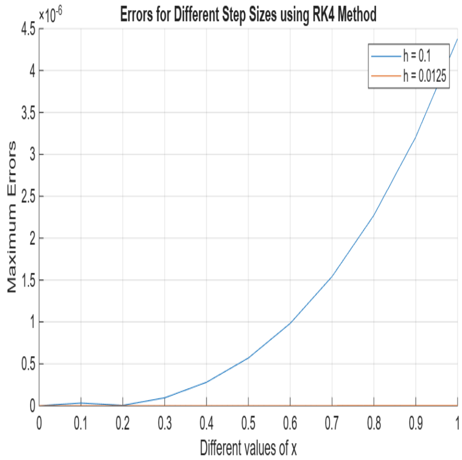

Figure 3: Error for different step sizes using RK−4 method.

Table 3: Numerical approximations and maximum errors for step size $h = 0.1000$

x_n	Euler's Method		Improved Euler Method		Runge-Kutta Method		Exact Solution
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	1.000000000000000	4.6548532e-03	0.995500000000000	1.5485319e-04	0.995345188541667	4.1730994e-08	0.995345146810673
0.2	0.991000000000000	8.1558720e-03	0.983157050000000	3.1292199e-04	0.982844212177690	8.4172674e-08	0.982844128005016
0.3	0.975180000000000	1.0342456e-02	0.965313070865000	4.7552683e-04	0.964837672389691	1.2835878e-07	0.964837544030914
0.4	0.954924600000000	1.1142257e-02	0.944426301227244	6.4395778e-04	0.943782519332497	1.7588494e-07	0.943782343447557
0.5	0.932727616000000	1.0574996e-02	0.922971543973245	8.1892401e-04	0.922152849037980	2.2907287e-07	0.922152619965111
0.6	0.911091235200000	8.7487427e-03	0.903342566370677	1.0000739e-03	0.902342783454897	2.9098458e-07	0.902342492470318
0.7	0.892425761088000	5.8490990e-03	0.887762318945961	1.1856568e-03	0.886577027348922	3.6523474e-07	0.886576662114187
0.8	0.878955957811840	2.1224557e-03	0.878205879518063	1.3723774e-03	0.876833957705618	4.5559018e-07	0.876833502115441
0.9	0.872639481186893	2.1449740e-03	0.876339920925292	1.5554657e-03	0.874785020609743	5.6539229e-07	0.874784455217451
1.0	0.875101927880072	6.6502728e-03	0.883481158081553	1.7289574e-03	0.881752897585865	6.9688031e-07	0.881752200705557

Table 4: Numerical approximations and maximum errors for step size $h = 0.0125$

x_n	Euler's Method		Improved Euler Method		Runge-Kutta Method		Exact Solution
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	0.995905928465374	5.6078165e-04	0.995347729560179	2.5827495e-06	0.995345146820878	1.0205614e-11	0.995345146810673
0.2	0.983815752799133	9.7162479e-04	0.982849306444708	5.1784397e-06	0.982844128025638	2.0622060e-11	0.982844128005016
0.3	0.966053552999778	1.2160090e-03	0.964845346047908	7.8020170e-06	0.964837544062428	3.1514014e-11	0.964837544030914
0.4	0.945071027071114	1.2886836e-03	0.943792811768270	1.0468321e-05	0.943782343490809	4.3252513e-11	0.943782343447557
0.5	0.923348310576836	1.1956906e-03	0.922165803850716	1.3183886e-05	0.922152620021459	5.6348037e-11	0.922152619965111
0.6	0.903295968231451	9.5347576e-04	0.902358432485576	1.5940015e-05	0.902342492541770	7.1452178e-11	0.902342492470318
0.7	0.887163863384730	5.8720127e-04	0.886595370262350	1.8708148e-05	0.886576662203499	8.9312446e-11	0.886576662114187
0.8	0.876961947128763	1.2844501e-04	0.876854940311994	2.1438197e-05	0.876833502226121	1.1068035e-10	0.876833502115441
0.9	0.874396979182001	3.8747604e-04	0.874808515326158	2.4060109e-05	0.874784455353636	1.3618462e-10	0.874784455217451
1.0	0.880827902787478	9.2429792e-04	0.881778689159943	2.6488454e-05	0.881752200871743	1.6618562e-10	0.881752200705557

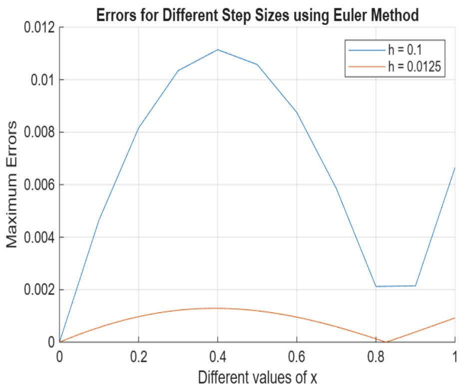

Figure 4: Error for different step sizes using Euler method.

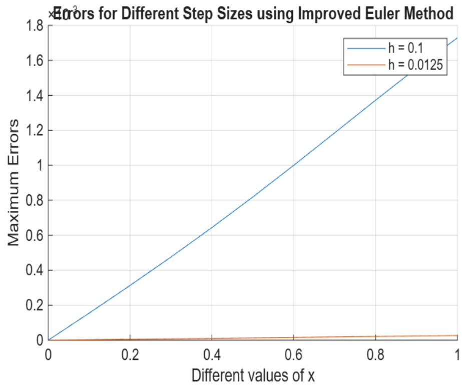

Figure 5: Error for different step sizes using Improved Euler method.

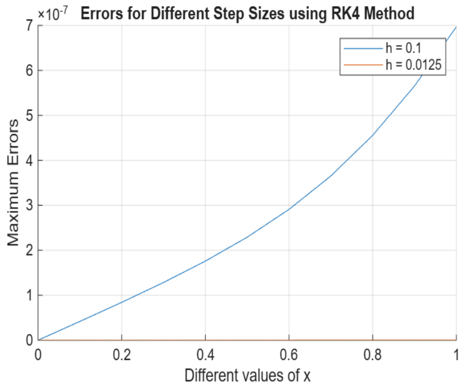

Figure 6: Error for different step sizes using RK−4 method.

Problem 3: We consider the initial value problem $y' = xy - y^2, y(0) = 1$ on the interval $0 \leq x \leq 1$ [10].The exact solution of the given problem is $y(x) = \frac{2e^{\frac{x^2}{2}}}{\sqrt{2\pi}\text{erfi}(\frac{x}{\sqrt{2}})+2}$.

4 Error Analysis

All the numerical methods involve some error.The primary goal of any numerical method is to keep the error as small as possible. At the same time, it is important to know about the bounds of error to have confidence on the results. There are two types of error in numerical methods: Truncation and round-off. Truncation errors arise when calculation is approximated instead of exact calculation. Round-off errors occur because computers can only store numbers with limited precision. These errors can be accumulated through successive calculations and may grow large enough to affect the final result. When small initial errors amplify and dominate the outcome, the numerical method is considered unstable. In both cases, the connection between the true value and its numerical approximation can be expressed as: True value - Approximated value = Error. The accuracy of the solution also depends on how small we choose the step size h. The maximum error defined by

$$e_{\max} = \max_{1 \leq n \leq N} |\, y(x_n) - y_n \,| \,.$$

where $y(x_n)$ denotes the approximate solution and y_n denotes the exact solution [10].

Table 5: Numerical approximations and maximum errors for step size $h = 0.1$

x_n	Euler's Method		Improved Euler's Method		Runge-Kutta Method		Exact Solution
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	0.900000000000000	1.3509128e-02	0.914000000000000	4.9087210e-04	0.913508932020453	1.9587833e-07	0.913509127898782
0.2	0.828000000000000	2.1218519e-02	0.849949762415992	7.3124371e-04	0.849218171060454	3.4764199e-07	0.849218518702443
0.3	0.776001600000000	2.5821798e-02	0.802671155116695	8.4775716e-04	0.801822944861821	4.5309578e-07	0.801823397957602
0.4	0.739063799679744	2.8719786e-02	0.768686804698807	9.0321854e-04	0.767783062126026	5.2403348e-07	0.767783586159507
0.5	0.714004821667228	3.0684879e-02	0.745619594120198	9.2989364e-04	0.744689128246402	5.7223223e-07	0.744689700478634
0.6	0.698724774214184	3.2163629e-02	0.731833146273825	9.4474350e-04	0.730887796150559	6.0662795e-07	0.730888402778509
0.7	0.691826629656969	3.3424670e-02	0.726208063309538	9.5676404e-04	0.725250665872579	6.3339952e-07	0.725251299272098
0.8	0.692392085182705	3.4635001e-02	0.727997757390170	9.7067117e-04	0.727026429582163	6.5663549e-07	0.727027086217658
0.9	0.699842772034956	3.5900817e-02	0.736732400101912	9.8881156e-04	0.735742909580024	6.7896493e-07	0.735743588544958
1.0	0.713850630961145	3.7289721e-02	0.752152528534882	1.0121766e-03	0.751139649932897	7.0202509e-07	0.751140351957987

Table 6: Numerical approximations and maximum errors for step size $h = 0.0125$

x_n	Euler's Method		Improved Euler Method		Runge-Kutta Method		Exact Solution
	y_n	E_n	y_n	E_n	y_n	E_n	$y(x)$
0.0	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000	0.0000000e+00	1.000000000000000
0.1	0.912023644337269	1.4854836e-03	0.913516675742420	7.5478436e-06	0.913509127886991	1.1790791e-11	0.913509127898782
0.2	0.846835976864134	2.3825418e-03	0.849229841166469	1.1322464e-05	0.849218518667959	3.4484304e-11	0.849218518702443
0.3	0.798877481268590	2.9459167e-03	0.801836608273060	1.3210315e-05	0.801823397902126	5.5476734e-11	0.801823397957602
0.4	0.764466294181218	3.3172920e-03	0.767797741288799	1.4155129e-05	0.767783586087147	7.2359674e-11	0.767783586159507
0.5	0.741110671096052	3.5790294e-03	0.744704347087938	1.4646609e-05	0.744689700393056	8.5577212e-11	0.744689700478634
0.6	0.727107563583767	3.7808392e-03	0.730903347844979	1.4945066e-05	0.730888402682344	9.6164632e-11	0.730888402778509
0.7	0.721297592738653	3.9537065e-03	0.725266489521617	1.5190250e-05	0.725251299167009	1.0508916e-10	0.725251299272098
0.8	0.722909634547887	4.1174517e-03	0.727042542546353	1.5456329e-05	0.727027086104554	1.1310353e-10	0.727027086217658
0.9	0.731458648526063	4.2849400e-03	0.735759369135689	1.5780591e-05	0.735743588424207	1.2075085e-10	0.735743588544958
1.0	0.746675912201788	4.4644398e-03	0.751156530721907	1.6178764e-05	0.751140351829581	1.2840584e-10	0.751140351957987

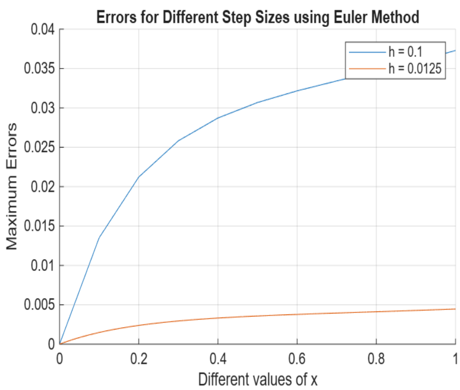

Figure 7: Error for different step sizes using Euler method.

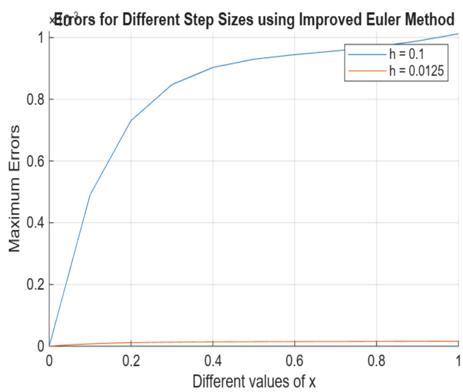

Figure 8: Error for different step sizes using Improved Euler method.

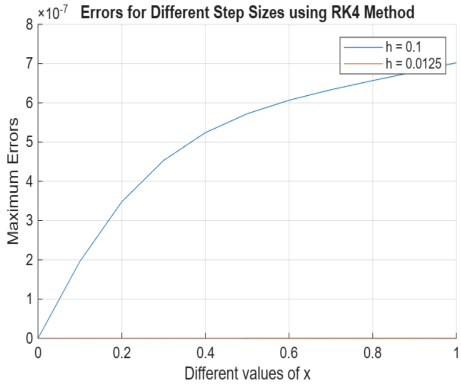

Figure 9: Error for different step sizes using RK−4 method.

5 Convergence, Consistency and Stability of Numerical Methods

Numerical methods for solving ordinary differential equations(ODEs) are examined not only on the basis of capacity of finding approximate solutions, but also considered the other fundamental qualities that assure the reliability of these solutions. Convergence, consistency and stability are the key features and are the necessary conditions for numerical methods. A numerical method is said to be convergent if

$$\lim_{h\to 0} \max_{0\le n\le N} |y(x_n) - y_n| = 0,$$

where $y(x_n)$ denotes the approximate solution and y_n denotes the exact solution [2].

Also, a numerical method is said to be consistent with the differential equation it approximates if its local truncation error approaches zero as the step size h tends to zero. Mathematically,

$$\frac{y_{n+1} - y_n}{h} \longrightarrow \frac{dy}{dx} \quad \text{as } h \to 0.$$

Again the numerical method is said to be stable if a slight change in the initial conditions do not result in an unbounded error in the numerical solution over time [3].

When consistency and stability are ensured, the order of convergence provides a quantitative measure of how rapidly the numerical solution approaches the exact solution as the step size decreases. In this context, Euler method exhibits first-order convergence, $O(h)$, meaning the error reduces approximately by half when the step size is halved. The Improved Euler method (Heun's method) demonstrates second-order convergence, $O(h^2)$ and the classical fourth-order Runge-Kutta method (RK−4) achieves fourth-order convergence, $O(h^4)$, offering significantly higher accuracy for the same step size [2, 10].

6 Sensitivity analysis for each method with different grid specification

The sensitivity analysis for the numerical methods was conducted by evaluating the maximum error for various step sizes h across three ODEs, where the first problem has a closed-form elementary solution and the second and third problems involve ODEs lack elementary form solutions. The results, plotted on log–log scales, reveal that the Euler method exhibits the largest errors and first-order convergence. The Improved Euler method shows better accuracy with second-order convergence, reflected in a steeper slope on the log–log plots, while the RK−4 method achieves the highest accuracy with errors several orders of magnitude smaller, demonstrating fourth-order convergence. We note that this relative performance pattern remains consistent even for ODEs lacking closed-form solutions, with RK−4 remaining the most accurate, followed by Improved Euler and then RK−4, and the maximum error consistently decreases as the step size decreases. Overall, the analysis confirms that higher-order methods are more accurate and

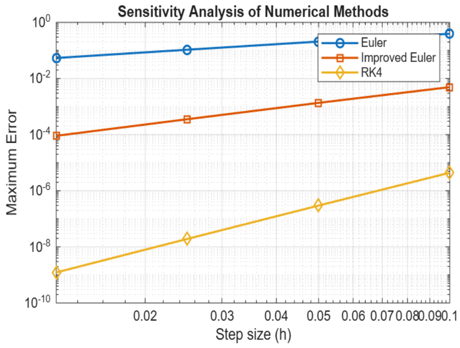

Figure 10: Sensitivity analysis of Problem 1.

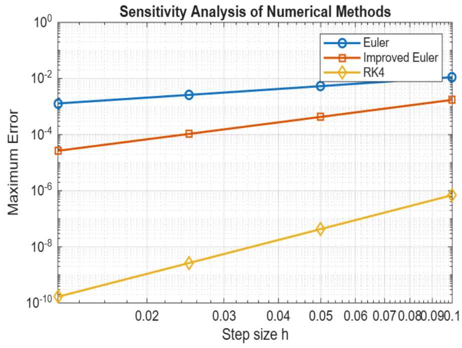

Figure 11: Sensitivity analysis of Problem 2

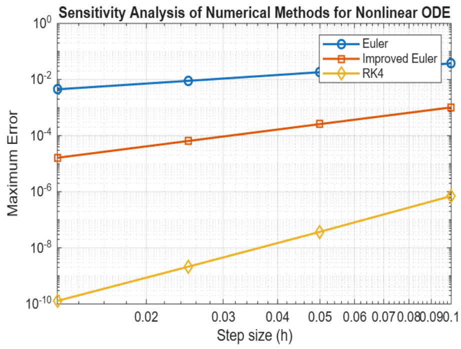

Figure 12: Sensitivity analysis of Problem 3.

less sensitive to step size changes than Euler method, with RK-4 emerging as the most reliable choice for problems with both elementary and non-elementary form solutions.

7 Discussion

In this work, we analyzed two linear ODEs and one nonlinear ODE. Only the first ODE has a closed-form solution i.e., solution is in terms of elementary function, while the other two lack closed-form solutions. For the latter, we used a special function called the error function to find their analytical solutions so that comparison is possible. We applied three numerical methods- Euler method, Improved Euler method, and the fourth-order Runge-Kutta (RK-4) method-using two different step sizes ($h = 0.1, 0.0125$) and presented the results in Tables 1 to 6. The maximum errors for each step size were also calculated and shown graphically in Figures 1 to 9. The results show that as the step size decreases, the numerical solutions become closer to the exact solution, with errors reducing as the step size approaches zero. From these tables, we observe that the RK-4 method consistently provides the most accurate results across selected step sizes, rapidly converging to the exact solution. However, it is also evident that Euler method and Improved Euler method show progressive improvement as the step size decreases. While these two methods do not converge to the exact solution for larger step size $h = 0.1$. They gradually approach to the analytical solution for smaller step size $h = 0.0125$. For the ODE with a closed-form solution, we directly compared its exact solution with the numerical approximations obtained from the three numerical methods. The results showed that accuracy varied depending on the step size. The Euler method had the highest error, improved Euler method is slightly improving but RK-4 method produced the most accurate results with the least error. To strengthen the analysis, we introduced new linear ODEs without a closed-form solution (Problems 2 and 3). For both the problems, the error function(special function) was used to obtain exact solutions, ensuring consistency in our comparison of numerical methods.

Further, by analyzing the errors from table and graphical representations, we confirmed that when a closed-form solution is available, numerical methods serve as a verification tool, ensuring that computational approximations align with the exact solution. However, when closed-form solutions do not exist, numerical methods become essential to provide an effective way to approximate the solution with controlled accuracy. The error analysis confirms that the accuracy of the methods follows this order: RK−4 > Improved Euler > Euler.

8 Conclusion

Most of the studies only compared the numerical solutions of ODEs with the exact solutions when the closed-form solutions exist and ignore the case when the closed form solutions are not available. In this work, using Euler, Improved Euler, and RK−4 method, we compare the numerical results with the exact one even when closed form solution is not available. We also discuss on convergence, consistency, stability and sensitivity analysis of the method used. Unlike previous works that focused only on ODEs with closed-form solutions, this study covers both the cases that is with closed form solution available and the absence of closed form solution. This study demonstrates that the RK−4 method is generally superior to the others and consistently produced the most accurate results, even when solutions interms of elementary function are not available. The solution obtained using this method rapidly approaches the analytical solution, with significantly lower error, compared to other two methods. The sensitivity analysis shows Euler is the most step-size sensitive, while RK−4 is the least, consistently giving the highest accuracy. Hence for ODEs which do have closed form solution, there is no any problem but for the ODEs which do not have closed-form solutions, numerical methods become indispensable, and among these, the RK−4 method proved to be the most accurate.

9 Acknowledgment

The first author acknowledges University Grant Commission for small RDI support SRDIG −81/82−S&T−04 and Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal.

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