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Statistical Convergence in Ordered Bicomplex Valued Metric Spaces

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Abstract: *This article presents the fundamental properties of bicomplex numbers and explores the partial order relations defined on them. It further develops bicomplex valued metric spaces based on these partial orderings and introduces a few theorems related to statistical convergence within the framework of bicomplex numbers.*

Keywords: Bicomplex valued metric space, Natural density, Partial order, Statistical convergence.

1 Introduction

The idea of statistical convergence was given by Fast [4] and independently developed by Schoenberg [13] for sequences of real and complex numbers. Later, several mathematicians, including Fridy [5], Salát [12], Tripathy & Nath [16], Tripathy & Sen [17], and Tripathy [18] extended this concept and explored its relationship with summability theory. The study of bicomplex numbers has also attracted considerable attention over the years, with notable contributions from Rochan & Shapiro [10], Sager & Sagir [11], Segre [14], Srivastava & Srivastava [15], and Wagh [19].

Das et al, [3] and many others studied the concept of statistical convergence of bicomplex valued metric spaces. In this work, we shall discuss the behavior of statistically convergent and statistically Cauchy sequences formulated within these spaces.

The concept of bicomplex numbers is introduced by Segre [14].

Definition 1.1. A bicomplex number is defined as $\gamma = a + ib + jc + ijd$, where $a, b, c, d \in \mathbb{C}_0$ and where i and j are such that $i^2 = j^2 = -1$ and $ij = ji = k$.

The collection of bicomplex numbers, denoted by $\mathbb{C}_2$ is described as follows:

$$\mathbb{C}_2 = \{\gamma : \gamma = a + ib + jc + ijd, \text{ where } a, b, c, d \in \mathbb{C}_0\}$$

i.e.

$$\mathbb{C}_2 = \{\gamma : \gamma = w + jz,\ w, z \in \mathbb{C}_1\}, \text{ where } w = a + ib \text{ and } z = c + id.$$

The bicomplex system contains four idempotent elements: $0, 1, e_1$ and e_2,
where $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$.
These elements are nontrivial and satisfy the relations $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
Let $\gamma = w + jz$ admits a unique decomposition in the idempotent basis given as.

$$\gamma = w + jz = (w - iz)e_1 + (w + iz)e_2 = \mu_1 e_1 + \mu_2 e_2,$$

$$\text{where } \mu_1 = w - iz \text{ and } \mu_2 = w + iz.$$

Definition 1.2. For $\gamma = w + jz \in \mathbb{C}_2$, the norm on $\mathbb{C}_2$ is defined as $\|\gamma\|_{\mathbb{C}_2} = \sqrt{|w|^2 + |z|^2}$

The multiplication of two bicomplex numbers γ and η satisfies the following inequality:

$$\|\gamma\eta\|_{\mathbb{C}_2} \leq \sqrt{2}\,\|\gamma\|_{\mathbb{C}_2}\|\eta\|_{\mathbb{C}_2}.$$

$\mathbb{C}_2$, together with the norm defined above, forms a generalized algebra. Since $\mathbb{C}_2 \simeq \mathbb{C}_0^4$ and $\mathbb{C}_0^4$ is complete with respect to the usual matrix, norm $\mathbb{C}_2$ forms a generalized Banach algebra. The bicomplex number $\gamma = w + jz$ is called singular if $|w^2 + z^2| = 0$. Recently, Jaiswal et al. [6, 7] & Parajuli et al. [8, 9] studied various aspects of the algebraic feature and topological structures of bicomplex numbers along with the sequence spaces built upon them.

2 Definition and Preliminaries

Before presenting the main results, we first outline the essential concepts and definitions that form the foundation of this research.

2.1 Partial Order relation

Definition 2.1. (Azam, [1]) The i partial order relation $\preccurlyeq_i$ on $\mathbb{C}_1$ can be described as follows: For $w, z \in \mathbb{C}_1, w \preccurlyeq_i z$, the real part of w is less than or equal to the real part of z, and the imaginary part of w is less than or equal to the imaginary part of z. (i.e. $Re(w) \leq Re(z)$ and $Im(w) \leq Im(z)$.)

Definition 2.2. Let $\gamma_1, \gamma_2 \in \mathbb{C}_2$, $\gamma_1 = w + jz$ and $\gamma_2 = w^* + jz^*$. The j partial order relation $\preccurlyeq_j$ on $\mathbb{C}_2$ is given by $\gamma_1 \preccurlyeq_j \gamma_2$ iff $w \preccurlyeq_i w^*$ and $z \preccurlyeq_i z^*$ i.e. $\gamma \preccurlyeq_j \eta$ if one of the following conditions is satisfied:

i. $w = w^*$ and $z = z^*$,

ii. $w \preccurlyeq_i w^*$ and $z = z^*$,

iii. $w = w^*$ and $z \preccurlyeq_i z^*$,

iv. $w \preccurlyeq_i w^*$ and $z \preccurlyeq_i z^*$,

In particular, we denote $\gamma \prec_j \eta$ when $\gamma \preccurlyeq_j \eta$ and $\gamma \neq \eta$.
when one of conditions (ii), (iii) and (iv) holds. Moreover we use $\gamma \preccurlyeq_j \eta$ precisely when condition (iv) alone is fulfilled.
For any pair of bicomplex numbers $\gamma, \eta \in \mathbb{C}_2$, the following inequalities can be established.

1. $\gamma \preccurlyeq_j \eta \implies \|\gamma\|_{\mathbb{C}_2} \leq \|\eta\|_{\mathbb{C}_2}$.
2. $\|\gamma + \eta\|_{\mathbb{C}_2} \leq \|\gamma\|_{\mathbb{C}_2} + \|\eta\|_{\mathbb{C}_2}$.

2.2 Bi-complex valued metric space

Choi et al. [2] defined a bicomplex valued metric space as follows.

Definition 2.3. [2] Let $d : Y \times Y \to \mathbb{C}_2$ be a mapping, where $Y \subseteq \mathbb{C}_2$.
The function d is said to be a bicomplex valued metric on Y relative to the j partial order when, for every $x_1, x_2, x_3 \in Y$, the following properties are satisfied.

i. $0 \preccurlyeq_j d(x_1, x_2)$.

ii. $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

iii. $d(x_1, x_2) = d(x_2, x_1)$.

iv. $d(x_1, x_2) \preccurlyeq_j d(x_1, x_3) + d(x_3, x_2)$.

It is represented by (Y, d_j).
The following lemma establishes a basic distance inequality in bicomplex valued metric spaces.

Lemma 2.1. *Let X be metric space with a bicomplex valued metric space d_j. Then, for all $\gamma_1, \eta_1, \gamma_2, \eta_2 \in \mathbb{C}_2$, we have*

$$d(\gamma_1, \eta_1) - d(\gamma_2, \eta_2) \preccurlyeq_j d(\gamma_1, \gamma_2) + d(\eta_1, \eta_2).$$

Proof. Applying the triangle inequality for the bicomplex metric d_j, it follows that

$$d(\gamma_1, \eta_1) \preccurlyeq_j d(\gamma_1, \gamma_2) + d(\gamma_2, \eta_2) + d(\eta_2, \eta_1).$$

Subtracting $d(\gamma_2, \eta_2)$ from both sides, we get

$$d(\gamma_1, \eta_1) - d(\gamma_2, \eta_2) \preccurlyeq_j d(\gamma_1, \gamma_2) + d(\eta_2, \eta_1).$$

Since $d(\eta_2, \eta_1) = d(\eta_1, \eta_2)$, it follows that

$$d(\gamma_1, \eta_1) - d(\gamma_2, \eta_2) \preccurlyeq_j d(\gamma_1, \gamma_2) + d(\eta_1, \eta_2).$$

Hence the result. □

2.3 Statistical convergence

The concept of natural density of sets of natural numbers forms the foundation for statistical convergence.

Definition 2.4. A subset $A \subseteq \mathbb{N}$ is known as natural density, represented by $\delta(A)$ and is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k),$$

where χ_A denotes the characteristic function of the set A.

Definition 2.5. Two sequences $(\mathbf{x}_k)$ and $(\mathbf{y}_k)$ are known as equal for almost all indices k if the set of indices where they differ have natural density zero, that is,

$$\delta(\{k \in \mathbb{N} : \mathbf{x}_k \neq \mathbf{y}_k\}) = 0.$$

Definition 2.6. A sequence (γ_k) in $\mathbb{C}_2$ is statistically convergent to γ (i.e $\gamma \in \mathbb{C}_2$), if, for every $\epsilon > 0$,

$$\delta\big(\{k \in \mathbb{N} : \|\gamma_k - \gamma\|_{\mathbb{C}_2} \geq \epsilon\}\big) = 0.$$

This is denoted by $stat - \lim \gamma_k = \gamma$.

Definition 2.7. Consider a bicomplex valued metric space (X, d_j) and a sequence (γ_k) in X. We say the sequence (γ_k) statistically convergent to a point $\gamma \in X$ whenever, for every $0 \prec_j \epsilon \in \mathbb{C}_2$, the natural density

$$\delta\big(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \succcurlyeq_j \epsilon\}\big) = 0.$$

This convergence is represented by stat-lim $\gamma_k = \gamma$.

Definition 2.8. In a bicomplex valued metric space (X, d_j), a sequence (γ_k) is known as statistically Cauchy sequence if, for every $0 \prec_j \epsilon \in X$,

$$\delta\big(\{k \in \mathbb{N} : d(\gamma_k, \gamma_m) \succcurlyeq_j \epsilon\}\big) = 0.$$

3 Main results

This section discusses various results concerning statistical convergence in a bicomplex valued metric space. The following result shows that the statistical convergence of sequences in a bicomplex metric space preserves the convergence of their mutual distances.

Lemma 3.1. *If a sequence (γ_k) in a bicomplex valued metric space (X, d_j) is statistically convergent, then the sequence of distance $\big(d(\gamma_k, \gamma)\big)$ statistically converges to 0 under the Euclidean norm on $\mathbb{C}_2$.*

Proof. Let (γ_k) be statistically convergent to γ in a bicomplex valued metric space (X, d_j). Then for every $\varepsilon \succ_j 0$, we have

$$\delta\big(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \not\succ_j \varepsilon\}\big) = 0$$

This implies $\delta\big(\{k \in \mathbb{N} : \|d(\gamma_k, \gamma)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}\big) = 0$.
Let $\varepsilon' = \|\varepsilon\|_{\mathbb{C}_2} > 0$.
Then $\delta\big(\{k \in \mathbb{N} : \|d(\gamma_k, \gamma)\|_{\mathbb{C}_2} \geq \varepsilon'\}\big) = 0$.
Hence, the sequence $(d(\gamma_k, \gamma))$ is statistically convergent to 0 under the Euclidean norm on $\mathbb{C}_2$, and is denoted by $\text{stat} - \lim d(\gamma_k, \gamma) = 0$. □

Theorem 3.2. *In a bicomplex valued metric space (X, d_j) if $(\gamma_k) \to \gamma$ and $(\eta_k) \to \eta$ are statistically convergent, then $\big(d(\gamma_k, \eta_k)\big) \to d(\gamma, \eta)$ statistically convergent to the Euclidean norm in $\mathbb{C}_2$.*

Proof. Using the characteristics of the bicomplex metric, it follows that

$$\{k \in \mathbb{N} : d(\gamma_k, \eta_k) - d(\gamma, \eta) \not\succ_j \varepsilon\} \subseteq \{k \in \mathbb{N} : d(\gamma_k, \gamma) \not\succ_j \varepsilon\} \cup \{k \in \mathbb{N} : d(\eta_k, \eta) \not\succ_j \varepsilon\}$$

Hence,

$$\delta\big(\{k \in \mathbb{N} : d(\gamma_k, \eta_k) - d(\gamma, \eta) \not\succ_j \varepsilon\}\big) \leq \delta\big(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \not\succ_j \varepsilon\}\big) + \delta\big(\{k \in \mathbb{N} : d(\eta_k, \eta) \not\succ_j \varepsilon\}\big)$$

Since both sequences (γ_k) and (η_k) are statistically convergent, the right-hand side is zero. Thus,

$$\delta\big(\{k \in \mathbb{N} : d(\gamma_k, \eta_k) - d(\gamma, \eta) \not\succ_j \varepsilon\}\big) = 0$$

Which implies that

$$\Rightarrow \delta\big(\{k \in \mathbb{N} : \|d(\gamma_k, \eta_k) - d(\gamma, \eta)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}\big) = 0. \text{ Hence the result}$$

□

The following statements are presented without proof.

Corollary 3.3. *Consider a bicomplex valued metric space (X, d_j), an element $\gamma, \eta \in X$. If (γ_k) is statistically convergent to both γ and η, then $\gamma = \eta$.*

This result states the uniqueness of statistical limits in a bicomplex valued metric space.

Lemma 3.4. *Let (X, d_j) be a metric space whose values lie in the bicomplex numbers $\mathbb{C}_2$, and define the distance between elements γ_k, γ as*

$$d(\gamma_k, \gamma) = d_1(\gamma_k, \gamma) + id_2(\gamma_k, \gamma) + jd_3(\gamma_k, \gamma) + kd_4(\gamma_k, \gamma),$$

where $d_m(\gamma_k, \gamma)$ for $m = 1, 2, 3, 4$ are real valued metrics corresponding to each bicomplex component. Then a sequence (γ_k) in X is statistically Cauchy with respect to d if and only if each of the four real sequences $(d_m(\gamma_k, \gamma))$, $m = 1, 2, 3, 4$ is statistically Cauchy in the corresponding real-valued metric spaces.

Lemma 3.5. *Let a metric space (X, d_j) where the metric takes values in the bicomplex numbers and assume that*

$$d(\gamma_k, \gamma) = d_1(\gamma_k, \gamma) + id_2(\gamma_k, \gamma) + jd_3(\gamma_k, \gamma) + kd_4(\gamma_k, \gamma).$$

Then a sequence (γ_k), where

$$\gamma_k = x_1 + ix_2 + jx_3 + kx_4, \quad x_1, x_2, x_3, x_4 \in \mathbb{C}_0, (ij = k)$$

is statistically convergent(statistically cauchy) in (X, d_j) iff (x_j) is statistically convergent (statistically cauchy) in the real valued metric space $(\mathbb{C}_2, d_j)$, $j = 1, 2, 3, 4$.
Based on the above, we can establish the following results.

Theorem 3.6. *If the sequences* (γ_k) *and* (η_k) *are convergent statistically in a bicomplex valued metric space* (X, d_j) *and if* $\|d_1(\gamma_k, \eta_k)\| \leq \|d(\gamma_k, \eta_k)\|$ *for all* $k \in \mathbb{N}$*, then the sequence* $(d_1(\gamma_k, \eta_k))$ *is also convergent statistically under the Euclidean norm in* $\mathbb{C}_2$*.*

Proof. Using Corollary 3.3, for all $\varepsilon \succcurlyeq_j 0$ and $k, m \geq n_0$, we obtain

$$\{k \in \mathbb{N}\colon d_1(\gamma_k, \eta_k) - d_1(\gamma_m, \eta_m) \succcurlyeq_j \varepsilon\} \subseteq \{k \in \mathbb{N}\colon d_1(\gamma_k, \gamma_m) \succcurlyeq_j \varepsilon\} \cup \{k \in \mathbb{N}\colon d_1(\eta_k, \eta_m) \succcurlyeq_j \varepsilon\}$$

$$\{k \in \mathbb{N}\colon \|d_1(\gamma_k, \eta_k) - d_1(\gamma_m, \eta_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}$$
$$\leq \{k \in \mathbb{N}\colon \|d_1(\gamma_k, \gamma_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\} \cup \{k \in \mathbb{N}\colon \|d_1(\eta_k, \eta_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}$$
$$\Rightarrow \delta\big(\{k \in \mathbb{N}\colon \|d_1(\gamma_k, \eta_k) - d_1(\gamma_m, \eta_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}\big)$$
$$\leq \big(\{k \in \mathbb{N}\colon \|d(\gamma_k, \gamma_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}\big) \cup \big(\{k \in \mathbb{N}\colon \|d(\eta_k, \eta_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\}\big)$$
$$\Rightarrow \delta\{k \in \mathbb{N}\colon \|d_1(\gamma_k, \eta_k) - d_1(\gamma_m, \eta_m)\|_{\mathbb{C}_2} \geq \|\varepsilon\|_{\mathbb{C}_2}\} = 0.$$

Hence, $(d_1(\gamma_k, \eta_k))$ forms a Cauchy sequence of bicomplex numbers and therefore $(d_1(\gamma_k, \eta_k))$ is statistically convergent with respect to the Euclidean norm. This completes the proof. □

The following lemma states that statistical convergence to a point ensures statistical convergence of pairwise distances.

Lemma 3.7. *Let* $\delta(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \succcurlyeq_j \varepsilon\}) = 0$ *then this implies that* $\delta(\{k \in \mathbb{N} : d(\gamma_k, \gamma_m) \succcurlyeq_j \varepsilon\}) = 0$.

Proof.

$$\delta(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \succcurlyeq_j \varepsilon\}) = 0 \Rightarrow \delta(\{k \in \mathbb{N} : d(\gamma_k, \gamma) \preccurlyeq_j (\tfrac{\varepsilon}{2})\}) = 1$$
$$\left\{k \in \mathbb{N}\colon d(\gamma_k, \gamma) \prec_j \left(\frac{\epsilon}{2}\right)\right\} \subseteq \{k \in \mathbb{N}\colon d(\gamma_k, \gamma_m) \preccurlyeq_j \epsilon\}$$
$$\Rightarrow \delta(\{k \in \mathbb{N}\colon d(\gamma_k, \gamma_m) \preccurlyeq_j \epsilon\}) = 1$$
$$\Rightarrow \delta(\{k \in \mathbb{N}\colon d(\gamma_k, \gamma_m) \succcurlyeq_j \epsilon\}) = 0. \text{ Hence the result.}$$

□

The converse is generally not true. Which can be shown in the following example.

Example 3.1. Let $X = (0, 1 + i + j + k)$ with the metric

$$d(\gamma, \eta) = (i + j + 1)\|\gamma - \eta\|_{\mathbb{C}_2} \text{ for all } \gamma, \eta \in X.$$

Let (γ_k) be a sequence in X defined as

$$\gamma_k = \begin{cases} \frac{i+j+ij+1}{k} & \text{for } k = i^2, i \in \mathbb{N} \\ \frac{i+j+ij+1}{k^2} & \text{otherwise} \end{cases}$$

The sequence (γ_k) may be statistically Cauchy without being statistically convergent in X.

The next result applies to complete bicomplex valued metric spaces.

Lemma 3.8. *Let* (X, d_j) *be a complete bicomplex valued metric space, and let* (γ_k) *be a sequence in* X*. Then* (γ_k) *is statistically convergent iff it is a statistically Cauchy.*

In a similar manner, a corresponding theorem holds for statistically Cauchy sequences in (X, d_j).

Theorem 3.9. *Let (γ_k) be a sequence in a bicomplex valued metric space (X, d_j). Assume that*

$$\left\{k \in \mathbb{N} \colon \sum_{l=1}^{k} d(\gamma_l, \gamma_{l+1}) \succsim_j \epsilon\right\}$$

Has natural density zero, under the assumption (γ_k), forms a statistically Cauchy sequence in (X, d_j).

Proof. We have

$$\delta(\left\{k \in \mathbb{N} \colon \sum_{l=1}^{k} d(\gamma_l, \gamma_{l+1}) \succsim_j \epsilon\right\}) = 0.$$

It follows that for each component metric $d_j; j = 1, 2, 3, 4$

$$\Rightarrow \delta(\left\{k \in \mathbb{N} \colon \sum_{l=1}^{k} d_j(\gamma_l, \gamma_{l+1}) \geq \epsilon_j\right\}) = 0. \quad j = 1, 2, 3, 4.$$

Hence

$$\Rightarrow \delta(\{k \in \mathbb{N} \colon d_j(\gamma_k, \gamma_{k+1}) \geq \epsilon_j\}) = 0, \qquad j = 1, 2, 3, 4.$$

Therefore, the sequence (γ_k) is a statistically Cauchy (or statistically convergent) with respect to the real valued metric d_j and consequently it is statistically Cauchy (or statistically convergent) in the bicomplex valued metric spaces (X, d_j). *for* $j = 1, 2, 3, 4$.
Therefore, (γ_k) is a statistically Cauchy in the bicomplex valued metric space (X, d_j). □

The next theorem describes the relationship between the statistical convergence of a bicomplex sequence and the statistical convergence of its component sequences.

Theorem 3.10. *Let a sequence (γ_k) of bicomplex numbers in the bicomplex valued metric space (X, d_j), where each term is of the form $\gamma_k = u_{1k} + j\, u_{2k}$. If (γ_k) statistically converges to $\gamma = u_1 + ju_2 \in X$. Then the component sequences (u_{1k}) and (u_{2k}) are statistically converge to u_1 and u_2, respectively.*

Proof. Since (γ_k) is statistically convergent to γ, for every $0 \prec \epsilon = \varepsilon_1 + j\,\varepsilon_2 \in \mathbb{C}_2$. The statistical density satisfies

$$\delta\big(\{k \in \mathbb{N} \colon d_j(\gamma_k, \gamma) \succsim_j \epsilon\}\big) = \lim_{n\to\infty} \frac{1}{n} \big|\{k \in \mathbb{N} \colon d_j(\gamma_k, \gamma) \succsim_j \epsilon\}\big| = 0.$$

To establish the behavior of the component sequences, we analyze two separate types of metrics.
Case I. Suppose that $d_j(\gamma_k, \gamma) = |u_{1k} - u_1| + j\,|u_{2k} - u_2|$
Which can be rewritten as

$$d_j(\gamma_k, \gamma) = d_1(u_{1k}, u_1) + jd_1(u_{2k}, u_2),$$

Where $d_1(u_k, u) = |u_k - u|$.
This gives a real valued metric structure on $\mathbb{C}_1$.
From

$$\big\{k \in \mathbb{N} \colon d_j(\gamma_k, \gamma) \succsim_j \epsilon\big\} = \big\{k \in \mathbb{N} \colon |u_{1k} - u_1| + j\,|u_{2k} - u_2| \succsim_j (\varepsilon_1 + j\varepsilon_2)\}$$

We obtain

$$\delta\big(\big\{k \in \mathbb{N} \colon |u_{1k} - u_1| \geq |\varepsilon_1|\big\}\big) \leq \delta\big(\big\{k \in \mathbb{N} \colon d_j(\gamma_k, \gamma) \succsim_j \epsilon\big\}\big) = 0,$$

$$\Longrightarrow \delta(\{k \in \mathbb{N} \colon |u_{1k} - u_1| \geq |\varepsilon_1|\}) = 0.$$

By a similar argument, $\delta(\{k \in \mathbb{N} \colon |u_{2k} - u_2| \geq |\varepsilon_2|\}) = 0$.
Thus, both (u_{1k}) and (u_{2k}) are convergent statistically in real valued metric space on $\mathbb{C}_1$.

Case II. Now assume the metric

$$d_j(\gamma_k, \gamma) = (a_1 + ja_2)\|\gamma_k - \gamma\|_{\mathbb{C}_2}, \text{ where } 0 \prec a_1, a_2 \in \mathbb{C}_1(i),$$

Equivalently, $d_j(\gamma_k, \gamma) = a_1\|\gamma_k - \gamma\|_{\mathbb{C}_2} + j\, a_2\|\gamma_k - \gamma\|_{\mathbb{C}_2}$
or

$$d_j(\gamma_k, \gamma) = a_1 d_1(\gamma_k, \gamma) + j\, a_2 d_1(\gamma_k, \gamma),$$

Where $d_1(\gamma_k, \gamma) = \|\gamma_k - \gamma\|_{\mathbb{C}_2}$
Let a real valued metric on $\mathbb{C}_2$ be given, and assume that the sequence (γ_k) is statistically convergent to γ with respect to this metric. We can write

$$\gamma_k = (u_{1k}, u_{2k}) \text{ and } \gamma = (u_1, u_2)$$

We obtain

$$\|\gamma_k - \gamma\|_{\mathbb{C}_2} = \sqrt{(u_{1k} - u_1)^2 + (u_{2k} - u_2)^2} = \sqrt{d_2^2(u_{1k}, u_1) + d_2^2(u_{2k}, u_2)},$$

and

$$|u_{1k} - u_1| \leq \sqrt{(u_{1k} - u_1)^2 + (u_{2k} - u_2)^2},$$

Which shows that

$$\{k\colon |u_{1k} - u_1| \geq \varepsilon\} \subseteq \{k\colon d_1(\gamma_k, \gamma) \geq \varepsilon\}.$$

It follows that the sequence (u_{1k}) is statistically convergent in the associated real valued metric space. An analogous argument shows that (u_{2k}) also converges statistically in its corresponding real valued metric space. This completes the proof. ☐

The following result shows that statistical convergence of the component sequences of a bicomplex sequence ensures the existence of equivalent pointwise convergent subsequences for almost all indices.

Theorem 3.11. *Let (γ_k) be a sequence in the bicomplex metric space (X, d_j), where each term can be written as $\gamma_k = u_{1k} + j\, u_{2k}$. Suppose the component sequences u_{1k} and u_{2k} are statistically converging to u_1 and u_2, respectively, then one may choose new sequences u'_{1k} and u'_{2k} that coincide with u_{1k} and u_{2k} except on a set of indices of natural density zero, and the modified sequence u'_{1k} and u'_{2k} converges to u_1 and u_2 respectively.*

Proof. Let (u_{1k}) *and* (u_{2k}) be sequence in (X, d_j) and assume that they converge statistically to u_1 *and* u_2 respectively.
Then for all $\varepsilon \succ_j 0$ in $\mathcal{C}_0$, the following hold:

$$\delta(\{k \in \mathbb{N}\colon d(u_{1k}, u_1) \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N}\colon d(u_{1k}, u_1) \geq \varepsilon\}| = 0.$$

and

$$\delta(\{k \in \mathbb{N}\colon d(u_{2k}, u_2) \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N}\colon d(u_{2k}, u_2) \geq \varepsilon\}| = 0.$$

Consider a strictly increasing sequence of natural numbers (n_k) with the property that, whenever $n > n_k$, the associated requirement holds:

$$\frac{1}{n}\left|\left\{k \in \mathbb{N}\colon d(u_{1k}, u_1) \geq \frac{1}{2^k}\right\}\right| < \frac{1}{2^k}.$$

Define a sequence (z_{1k}) in $\mathcal{C}$ by

$$z_{1k} = \begin{cases} u_{1k} & \text{if } k \leq n_1 \\ u_{1k} & \text{if } d(u_{1k}, u_1) \geq \frac{1}{2^k} \\ u_1 & \text{otherwise} \end{cases}.$$

Then the sequence (z_{1k}) is convergent. Moreover,

$$\{k \in \mathbb{N} \colon u_{1k} = (z_{1k})\} \supseteq \{k \colon d_i(u_{1k}, u_1) \preccurlyeq_i \varepsilon\},$$

which implies that $u_{1k} = (z_{1k})$ for almost all k.
Similarly, we can construct a convergent sequence (z_{2k}) such that $u_{2k} = (z_{2k})$ for almost all k. □

Conclusion

This research examines the behavior of sequences with respect to statistical convergence in bicomplex valued metric spaces. Future work will explore the construction of statistically Cauchy sequences and examine completeness within such spaces.

Declaration of Competing Interests

The authors declare that they have no conflicts of interest.

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