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On Farthest Points

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Abstract: *A non-empty bounded subset T of a metric space (X, d) is said to be remotal (uniquely remotal) if for each $x \in X$ there exists at least one (exactly one) $t \in T$ such that $d(x, t) = \sup\{d(x, y) : y \in T\}$. Such a point t is called a farthest point of x in T. In this paper, we discuss remotal sets, uniquely remotal sets and the singleton property of uniquely remotal sets, thereby providing some partial affirmative answers to the hitherto unsolved farthest point problem: If every point of a normed linear space X admits a unique farthest point in the set T, must T be a singleton? The underlying spaces considered are metric spaces, convex metric spaces, and linear metric spaces.*

Keywords: Remotal set, Uniquely remotal set, Farthest point map, Convex metric space.

1 Introduction

Let (X, d) be a metric space and T a non-empty bounded subset of X. The set T is said to be **remotal (uniquely remotal)** if for each $x \in X$ there exists at least one (exactly one) $t \in T$ such that $d(x, t) = \sup\{d(x, y) : y \in T\} \equiv \delta(x, T)$. Such a point t is called a **farthest point** of x in T. The set-valued map $F_T : X \to 2^T \equiv$ the collection of all subsets of T, defined by

$$F_T(x) = \{t \in T : d(x, t) = \sup_{y \in T} d(x, y)\} \equiv F(x, T)$$

is called the **farthest point map (f.p.m.)** or **metric antiprojection**. If T is uniquely remotal i.e $F_T : X \to T$ is single-valued, we denote F_T simply by F. B. Jessen [4] initiated the study of farthest points in normed linear spaces. One of the most interesting, intriguing and hitherto unsolved problems in the theory of farthest points, known as the farthest point problem (f.p.p.), is: If every point of a normed linear space X admits a unique farthest point in the set T, must T be a singleton? This problem was perhaps first proposed by Jessen [4]. Motzkin et al. [7] analyzed the problem in Euclidean space R^n, whereas Klee [6] considered it in the context of Banach spaces of any dimension. Klee [6] proved that every compact uniquely remotal subset of a Banach space is a singleton. Since then, much progress has been made towards resolving the f.p.p. by several researchers. There are many partial affirmative answers to this problem, and some special cases in which the answer is negative (see e.g. [1]-[11], [13]-[16], [18]-[20] and references cited therein), but the problem remains unsolved.

The study of farthest points has attracted many researchers because of its applications in the study of extremal structures of sets, characterization of compact convex sets, deviation of sets, convex sets extensively used in programming, and in the study of geometry of Banach spaces (see e.g. [8], [12] and references cited therein). In this paper, we discuss remotal sets, uniquely remotal sets and the singleton property of uniquely remotal sets, thereby providing some partial affirmative answers to the f.p.p.. The underlying spaces considered are metric spaces, convex metric spaces, and linear metric spaces. The results proved here generalize and extend some of the results of [1], [2] and [13]-[15].

2 Preliminaries

In this section, we shall discuss some basic notations and definitions that are used in this paper.

2.1 Metric midpoint, x-compact, local max-sun, center, convex structure, almost a center

Let (X, d) be a metric space and $x, y, z \in X$. We say that $z \in X$ is a **metric midpoint** or (**a center**) of x and y if

$$d(x, z) = d(z, y) = \frac{1}{2}d(x, y).$$

A metric space is said to satisfy **metric midpoint property** if each pair of points of it has a metric midpoint. The set Q of rational numbers with usual metric $d(x, y) = |x - y|$ has metric midpoint property. $(Q, |.|)$ is not a convex metric space. It is neither a normed linear space nor a linear metric space over $\mathbb{R}$. Let (X, d) be a metric space, E a closed bounded subset of X and $x \in X$. We say that E is ***x-compact*** if $diam(E \backslash B_n(x)) \to 0$ as $n \to \infty$, where

$$B_n(x) = \{y \in X : d(y, x) \leq \delta(x, E) - \frac{1}{n}\}.$$

A subset M of a metric space (X, d) is called a **local max-sun**[1] at a point $x_o \in X$ if there is a farthest point $\hat{y} \in F_M(x_o)$ such that $\hat{y} \in F_M(x)$ for any point $x \in [x_o, \hat{y}, -) \equiv \{z \in X : d(x_o, \hat{y}) + d(\hat{y}, z) = d(x_o, z)\}$, the half ray starting from x_o and passing through $\hat{y}$.

The set M is called a local max-sun on a set U if M is a local max-sun at each point $x_o \in U$. In a non-singleton set E, we say that $F(c)$ is **isolated** if for all x in a neighbourhood of c on $(c, F(c)]$, $d(F(x), F(c)) > \gamma$ for some $\gamma > 0$. Here F is any single-valued function extracted from the multi-valued function $F(., E)$.

A **center** of a subset E of a metric space (X, d) is an element $c \in X$ such that $\delta(c, E) = \inf_{x \in X} \delta(x, E)$ i.e. $\sup_{e \in E} d(c, e) = \inf_{x \in X} \sup_{e \in E} d(x, e)$. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure**[17] on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$. A metric space (X, d) together with a convex structure W, denoted by (X, d, W), is called a **convex metric space**.

Let E be a closed remotal subset of a convex metric space (X, d, W) and $c \in X$. If $F(c) \in F(c, E)$ is such that $\delta(c, E) \leq \delta(x, E)$ for every $x \in (c, F(c)] \equiv \{W(c, F(c), \lambda) : \lambda \in (0, 1]\}$, then c is said to be **almost a center**[14] of E.

3 Methods

A natural trend in mathematical research is to refine the framework of known results and to examine which of the results available in Banach and Hilbert spaces survive in more general spaces. Most of the literature on farthest points is in Hilbert or Banach spaces. Extending the theory in more general spaces is a challenging one but some attempts have been made in this direction (see e.g. [9]-[11] and references cited therein). This paper is also a step in the same direction.

4 Main Results

In this section, we discuss some results on farthest points in metric spaces, convex metric spaces and linear metric spaces which are more general than Banach and Hilbert spaces.

Theorem 4.1 If E is a closed bounded subset of a complete metric space (X, d) and E is x-compact, then $\delta(x, E)$ is attained.

Proof: Let $\{e_n\} \subset E$ be such that $d(x, e_n) \to r = \delta(x, E)$. The sequence $\{e_n\}$ can be selected so that $d(x, e_n) \uparrow r$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ be such that $diam(E \backslash B_N(x)) < \epsilon$. Such an N exists because E is x-compact. Let $M \in \mathbb{N}$ be such that $d(x, e_n) > r - \frac{1}{M}$, then for $n, m > M$, we have $d(e_n, e_m) < \epsilon$ because

$e_n, e_m \in E \backslash B_N(x)$. This shows that $\{e_n\}$ is a Cauchy sequence in the complete metric space (X, d). Since E is complete, $e_n \to e \in E$ and $d(x, e) = \lim_{n \to \infty} d(x, e_n) = r$, i.e., $\delta(x, E)$ is attained by e.

Theorem 4.2: If a closed bounded set E is x-compact for some x in the complete metric space (X, d), then $\delta(x, E)$ is attained by a unique element $e \in E$.

Proof: Since E is x-compact, by the above theorem $\delta(x, E)$ is attained by some element $e \in E$. Suppose $\delta(x, E)$ is attained by two elements e and $e\prime$. We claim that $e = e\prime$. We know that $d(x, e) = d(x, e\prime) > \delta(x, E) - \frac{1}{N}$ for all $N \in \mathbb{N}$. This implies that $diam(E \backslash B_N(x)) \geq d(e, e\prime)$. Since E is x-compact, $diam(E \backslash B_n(x)) \to 0$. Therefore $d(e, e\prime) = 0$. Hence $e = e\prime$.

Corollary 4.1 : Let E be a closed bounded subset of a complete metric space (X, d). If E is x-compact for every $x \in X$, then E is uniquely remotal. For normed linear spaces, Theorems 4.1, and 4.2 were proved in [14].

Theorem 4.3: If M is a local max-sun on the entire metric space (X, d), then M is a singleton.

Proof: We fix arbitrary $x \in X$ and $y \in F_M(x)$. By assumption, such a y exists. Consider the set $Z = \{z \in X : d(x, y) = d(x, z) + d(z, y), y \in F_M(z)\}$. Since $x \in Z$, we have $Z \neq \phi$. It is easy to see that if $z \in Z$, then $y \in F_M(z)$. Given $u, v \in Z$, we write $u \leq v$ if

$$d(x, y) = d(x, u) + d(u, v) + d(v, y).$$

Note that if $u, v \in Z$ and $u \leq v$, then

$$d(x, v) = d(x, u) + d(u, v)$$

and

$$d(u, y) = d(u, v) + d(v, .y)$$

It is easy to see that $\leq$ is a partial order. By Zorn's lemma there exists a maximal chain C in Z. We set

$$d\prime = \sup_{z \in C} d(x, z).$$

By the definition of sup, there exists a sequence $\{z_k\} \subset C$ such that $d(x, z_k) \to d\prime$. We have $d(z_n, z_k) = |d(x, z_k) - d(x, z_n)| \to 0$ as $n, k \to \infty$ and hence $\{z_k\}$ is a Cauchy sequence. Since X is complete, $\{z_k\}$ converges to some $\hat{z} \in X$. It is easy to see that $\hat{z} \in Z$. If $\hat{z} = y$ then $M = \{y\}$ is a singleton, which is the required result. So assume that $\hat{z} \neq y$. By the hypothesis, there exists $y\prime \in F_M(\hat{z})$ and $w \in (\hat{z}, y\prime)$ such that $y\prime \in F_M(w)$. We claim that $y \in F_M(w)$. Indeed, $d(\hat{z}, y) = d(\hat{z}, y\prime) = \delta(\hat{z}, M)$.

$$\begin{aligned} d(w, y) &\geq d(\hat{z}, y) - d(w, \hat{z}) \\ &= d(\hat{z}, y\prime) - d(w, \hat{z}) \\ &= d(w, y\prime) \\ &= \delta(w, M) \geq d(w, y) \end{aligned}$$

which shows $d(w, y) = \delta(w, M)$, and so $y \in F_M(w)$. Next

$$\begin{aligned} d(x, w) + d(w, y) &\leq d(x, \hat{z}) + d(\hat{z}, w) + d(w, y) \\ &= d(x, \hat{z}) + d(\hat{z}, w) + d(w, y\prime) \\ &= d(x, \hat{z}) + d(\hat{z}, y\prime) \\ &= d(x, \hat{z}) + d(\hat{z}, y) \\ &= d(x, y). \end{aligned}$$

Therefore $w \in Z$. Since C is a maximal chain, we have $w \in C$. Moreover,

$$d(x, w) = d(x, \hat{z}) + d(\hat{z}, w) > d(x, \hat{z}) = d\prime.$$

But this is a contradiction to the definition of $d\prime$.

For normed linear spaces Theorem 4.3 was proved in [1]. The following result proved in [15] shows that a remotal subset with more than two points in a metric space is uniquely remotal whenever each of its non-trivial subsets is uniquely remotal.

Theorem 4.4 Let (X, d) be a metric space and T a remotal subset of X containing at least three points. Then T is uniquely remotal if every non-trivial subset of T is uniquely remotal.

Remark 4.1: If T is a subset of a metric space (X, d) satisfying midpoint property (or is a convex metric space or is a linear metric space) and T consists of only two points, then the conclusion of Theorem 4.4 is not true.

Let $T = \{u, v\}$, then each non-trivial subset (which are singleton) of T is uniquely remotal but T is not as $\frac{1}{2}d(u, v) = d(t_o, u) = d(t_o, v) = \delta(t_o, T)$, where t_o is mid-point of u and v. (If (X, d, W) is a convex metric space, then $\frac{1}{2}d(u, v) = d(W(u, v, \frac{1}{2}), u) = d(W(u, v, \frac{1}{2}), v) = \delta(W(u, v, \frac{1}{2}), T)$. If (X, d) is a linear metric space, then $d(\frac{u+v}{2}, u) = d(\frac{u+v}{2}, v) = \delta(\frac{u+v}{2}, T)$.)

Remark 4.2: The converse of Theorem 4.4 is equivalent to the f.p.p. in metric spaces satisfying midpoint property, in convex metric spaces and in linear metric spaces.

Theorem 4.5 : In a metric space (X, d) satisfying midpoint property (or in a convex metric space or in a linear metric space), every uniquely remotal set is a singleton if and only if every non-trivial subset of each uniquely remotal set is uniquely remotal.

Proof: Suppose every uniquely remotal set is a singleton. If T is uniquely remotal, then T is a singleton and so are all its non-trivial subsets.

Conversely, suppose that every non-trivial subset of each uniquely remotal set is uniquely remotal. If T is a uniquely remotal set and consists of at least two points u and v, then $T_o = \{u, v\} \subseteq T$ should be uniquely remotal. But

$$\delta(t_o, T_o) = d(t_o, u) = d(t_o, v) = \frac{1}{2}d(u, v)$$

shows that t_o (the midpoint of u and v)$\in X$ has two farthest points u and v in T_o, contradicting our assumption. (For a convex metric space (X, d, W) $d(W(u, v, \frac{1}{2}), T_o)) = d(W(u, v, \frac{1}{2}), u) = d(W(u, v, \frac{1}{2}), v) = \frac{1}{2}d(u, v)$. For a linear metric space X, $d(\frac{u+v}{2}, T_o) = d(\frac{u+v}{2}, u) = d(\frac{u+v}{2}, v) = \delta(\frac{u+v}{2}, T_o)$).

For normed linear spaces Theorem 4.4 was proved in [14] and for convex metric spaces in [15].

Theorem 4.6 Let E be a closed bounded remotal subset of a convex metric space (X, d, W) admitting a center c. If E is not a singleton, then for any single-valued selection F of $F(x, E)$ there exists a $\gamma > 0$ such that $d(F(x), F(c)) > \gamma$ for all x in a neighbourhood of c on $(c, F(c)) \equiv \{W(c, F(c), \lambda) : \lambda \in (0, 1)\}$.

Proof: Let F be a single-valued selection from $F(., E)$ and $x \in (c, F(c))$, then $x = W(c, F(c), t)$ for some $t \in (0, 1)$, and so

$$\begin{aligned} d(x, F(x)) &= d(W(c, F(c), t), F(x)) \\ &\leq t\, d(c, F(x)) + (1 - t)\, d(F(c), F(x)) \\ &\leq t\, d(c, F(c)) + (1 - t)\, d(F(c), F(x)) \\ &\leq t\, d(x, F(x)) + (1 - t)\, d(F(c), F(x)). \end{aligned}$$

This implies

$$d(F(c), F(x)) \geq d(x, F(x)) \geq d(c, F(c)) = r.$$

Thus γ can be chosen so that $\gamma \leq r$, the Chebyshev radius of E.
Therefore for the farthest point mapping to be continuous at c, the set E must be a singleton. This isolation result generalizes all known results relating the continuity of the farthest point mapping and singleton property of the uniquely remotal sets.

It may be remarked that Astaneh [2] proved this isolation result in inner product spaces, Astaneh proved that if c is center of closed bounded subset E in an inner product space and if E is a non-singleton and uniquely remotal, then a $\gamma > 0$ exists with the above isolation property. The above result generalizes this result in many ways, when our space can be any convex metric space (not a normed linear space or an inner product space) and our E need not be uniquely remotal, but only remotal.

Theorem 4.7 : Let E be a non-singleton closed bounded subset of a convex metric space (X, d, W) admitting a center c and γ be as in Theorem 4.6 . If the distance $\delta(c, E \setminus B(F(c), \gamma))$ is attained, then E can not be uniquely remotal.

Proof: Suppose E is uniquely remotal and c is center. Let $a \in (c, F(c)]$ be such that $d(F(x), F(c)) \geq \gamma$ for every $x \in (c, a]$. Let $\{x_n\} \in (c, a]$ be such that $\{x_n\} \to c$. By the above isolation result $F(x_n) \in E \setminus B(F(c), \gamma)$. Now

$$\delta(c, E) = \lim_{n\to\infty} \delta(x_n, E) = \lim_{n\to\infty} \delta(x_n, E \setminus B(F(c), \gamma)) = \delta(c, E \setminus B(F(c), \gamma)).$$

Consequently, since $\delta(c, E \setminus B(F(c), \gamma)$ is attained by the assumption, there exists $e \in E \setminus B(F(c), \gamma)$ such that $\delta(c, E) = d(c, e)$. Since $e \in E \setminus B(F(c), \gamma)$, $e \neq F(c)$ contradicting the assumption that E is uniquely remotal.

Corollary 4.2: Let E be a closed bounded subset of a convex metric space (X, d, W) admitting a center c and F be any single-valued mapping of $F(., E)$. If a sequence $\{x_n\} \in (c, F(c)]$ exists such that $x_n \to c$ and $\{F(x_n)\}$ converges, then E can not be uniquely remotal.

Proof: The proof is immediate since $F(x_n) \to e$ implies $e \in F(x, E)$.

Corollary 4.3 If a compact subset E with a center c, is uniquely remotal in a convex metric space (X, d, W), then E must be a singleton.
Proof: Assume E is not a singleton. Let γ be as above. Since E is compact, $E \setminus B(F(c), \gamma)$ is compact and hence remotal. That is $\delta(c, E \setminus B(F(c), \gamma))$ is attained. Consequently, E is not uniquely remotal by Theorem 4.7, contradicting our assumption. This completes the proof.

It may be remarked that the above proof uses unique remotality of E with respect to its center only i.e. for E to be singleton, E need not be uniquely remotal. In other words, we have

Theorem 4.8 Let E be a compact subset of a convex metric space (X, d, W), with a center. If E is uniquely remotal with respect to its center then E is a singleton.

Remark 4.3 : It is easy to prove that Theorems 4.6, 4.7, 4.8 and Corollaries 4.2, 4.3, which were proved in [13] for normed linear spaces, are still valid if c is almost a center rather than being a center.

5 Conclusion

In this paper, we have discussed some results concerning remotal sets, uniquely remotal sets and singleton property of uniquely remotal sets when the underlying spaces are metric spaces or convex metric spaces or linear metric spaces. The results provide some partial affirmative answers to the f.p.p. in these spaces. The proved results generalize and extend some of the results proved in [1, 2], [13]-[15] on the subject.

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