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Enumeration and Bijections of a Class of Labelled Plane Trees and Related Structures

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Abstract: *In this paper, we introduce and enumerate weakly labelled k-plane trees by number of vertices, occurrences of vertices of a certain type, root degree, label of the eldest child of the root, label of the youngest child of the root, number of leaves, forests and outdegree sequence. Bijections between the set of these trees and the sets of k-ary Husimi graphs, k-plane trees, plane trees where each internal vertex has a given number of leaf children and weakly labelled k-trees are also constructed.*

Keywords: Weakly labelled k-plane tree, k-ary Husimi graph, Weakly labelled k-tree.

1 Introduction and preliminary result

Plane trees (also called *ordered trees*) are rooted trees in which all subtrees are ordered [2]. They are one of the many structures enumerated by the Catalan number [17]. Most of these structures are listed in sequence A006013 of the online encyclopaedia [15]. The *degree* of a vertex v in a plane tree is the number of vertices incident to v. The arrangement of degrees of a tree is its *degree sequence.* A vertex of degree 1 is a *leaf* and a non-leaf vertex is an *internal vertex.* By a *forest*, we mean a collection of trees. The number of edges in a path from the root to a given vertex is the *level* of that vertex. The vertices that are incident to v but are on a lower level are the *children* of v and vertex v is their *parent.* Vertices which share a parent are said to be *siblings.* The child that appears at the far left is its *eldest child* with the child on the far right being the *youngest child.* The number of children of a vertex is its *outdegree* and an arrangement of outdegrees of a tree is the *outdegree sequence.* Note that for the root, outdegree and degree coincide. Plane trees have been enumerated by various authors according to number of vertices, leaves, root degree, level of a vertex [3], degree sequence [4], outdegree sequence [14, 16] and forests [16] among other parameters. Plane trees have been generalized by considering their block graphs [8, 11, 13] and assigning labels to the vertices [5, 6]. We introduce another generalization based on the latter.

Definition 1.1. *A* weakly labelled k-plane tree *is a plane tree whose root is unlabelled and all children of all internal vertices are given labels from the set $\{1, 2, \ldots, k\}$ such that the labels are non-decreasing from left to right.*

For an example of a weakly labelled k-plane tree, see Figure 1.

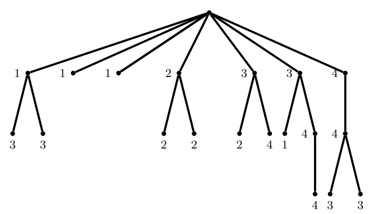

Figure 1: A weakly labelled 4-plane tree on 20 vertices.

The following theorem, which we prove in the next section, gives a formula counting these combinatorial structures.

Theorem 1.2. *There are*

$$\frac{1}{n}\binom{(k+1)n-2}{n-1} \tag{1}$$

weakly labelled k-plane trees on n vertices.

In this paper, we enumerate weakly labelled k-plane trees by number of vertices of a certain type, root degree, label of the eldest and youngest child of the root, number of leaves, associated forests and outdegree sequence in Section 2. In Section 3, we construct bijections between the set of weakly labelled k-plane trees and the sets of other combinatorial structures which we now describe. We remark that the results presented in this paper form part of the PhD thesis of the first author. The work was supervised by the second author.
Husimi graphs or block graphs have been studied by various authors such as [1, 7, 13] and recently by Kariuki and Okoth [8]. *Plane Husimi graphs* were introduced by Okoth in [11] as generalizations of plane trees. These are structures obtained when edges in plane trees are replaced by complete graphs such that the resultant structures are connected and cyclefree. In [13], Onyango, Okoth and Kasyoki studied the set of these structures and the set of k-ary Husimi graphs. They defined *k-ary Husimi graphs* as plane Husimi graphs whose vertices have at most outdegree k. The aforementioned authors obtained that the number of k-ary Husimi graphs on n vertices is given by (1), a formula which also counts weakly labelled k-plane trees on n vertices. A bijection between the sets of these combinatorial structures is obtained in Subsection 3.1.
Gu, Prodinger and Wagner, in [6], introduced and enumerated *k-plane trees* which are plane trees whose vertices are labelled with integers from the set $\{1, 2, 3, \ldots, k\}$ such that the sum of the labels of any two adjacent vertices is at most $k+1$. If the number of vertices is n and the root is labelled i, then there are

$$\frac{k+1-i}{kn+1-i}\binom{(k+1)n-i-1}{n-1}$$

such k-plane trees. Setting $i = k$, we see that

$$\frac{1}{k(n-1)+1}\binom{(k+1)(n-1)}{n-1} \tag{2}$$

counts k-plane trees on n vertices with root labelled k. A refinement of this formula has since been obtained by Okoth and Wagner [12] to take into consideration the number of occurrences of vertices of certain labels.
David Callan introduced the set of plane trees with n internal vertices and n leaves where each internal vertex has a single leaf and showed that such tree are enumerated by $\frac{1}{n+1}\binom{3n+1}{n}$, as recorded in sequence $A006013$ of [15]. A generalization of this result shows that the set of plane trees with $k(n-1)$ vertices where each internal vertex has $k-1$ leaves is enumerated by (1). In Subsection 3.2, we obtain a bijection between the set of weakly labelled k-plane trees and the set of plane trees where each internal vertex has $k-1$ leaves.

Nyariaro and Okoth, in [10], enumerated the set of *weakly labelled k-trees* on n vertices and constructed their bijections. These are labelled trees whose roots are unlabelled and all children of the root are labelled with a fixed integer i for $i \in \{1, 2, \ldots, k\}$ and the labels of all vertices sharing a parent are non-decreasing. These trees on $n+1$ vertices are also enumerated by (2). When $k = 2$, we obtain the set of weakly labelled 2-trees on $n+1$ vertices which is enumerated by

$$\frac{1}{2n+1}\binom{3n}{n}. \tag{3}$$

The same formula counts noncrossing trees and 2-plane trees among many other combinatorial structures as listed in [15, A001764]. In Subsection 3.3, we construct a bijection between the set of weakly labelled k-plane trees and the set of k-plane trees on n vertices with their roots labelled 1. We also construct a bijection between the set of weakly labelled k-trees and the set of k-plane trees with root labelled k, enumerated by (2).

2 Enumeration of weakly labelled k-plane trees

2.1 Occurrences of vertices of certain labels

Let $P_i(x, u_1, \ldots, u_k) = P_i$ be the multivariate generating function for weakly labelled k-plane trees whose root is labelled i, where x marks a vertex and u_j marks a vertex labelled j for $j = 1, \ldots, k$. Note that if the root is not labelled then any label can be assigned to it. Since $P_i(x, u_1, \ldots, u_k)$ consists of a root vertex and a sequence of subtrees rooted at vertices labelled 1, then those labelled 2, and so on, until we get a sequence of subtrees rooted at vertices labelled k. So, we have

$$P_i = x \cdot \frac{1}{1 - u_1 P_1(x)} \cdot \frac{1}{1 - u_2 P_2(x)} \cdots \frac{1}{1 - u_k P_k(x)}. \tag{4}$$

Since $P_1 = P_2 = \cdots = P_k$, then from (4), we obtain

$$P_i(x) = x \prod_{j=1}^{k} \left(1 - u_j P_i(x)\right)^{-1}.$$

By Lagrange inversion formula [16, Theorem 5.4.2], we get

$$\begin{aligned}
[x^n u_1^{a_1} \cdots u_k^{a_k}] P_i &= \frac{1}{n}[p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^{k} (1 - u_j p)^{-n} = \frac{1}{n}[p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^{k} \sum_{i \geq 0} \binom{-n}{i} (-u_j p)^i \\
&= \frac{1}{n}[p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^{k} \sum_{i \geq 0} \binom{n+i-1}{i} p^i u_j^i = \frac{1}{n}[p^{n-1}] p^{a_1 + \cdots + a_k} \prod_{j=1}^{k} \binom{n + a_j - 1}{a_j} \\
&= \frac{1}{n} \prod_{j=1}^{k} \binom{n + a_j - 1}{a_j}.
\end{aligned}$$

The last equality follows since $a_1 + \cdots + a_k = n - 1$, i.e., all non-root vertices are labelled with an integer in the set $\{1, 2, \ldots, k\}$. Thus, we have the following theorem.

Theorem 2.1. *The number of weakly labelled k-plane trees on n vertices such that there are a_j vertices labelled j is given by*

$$\frac{1}{n} \prod_{j=1}^{k} \binom{n + a_j - 1}{a_j}. \tag{5}$$

If we set $k = 1$ in (5), then $a_1 = n - 1$ and we find that there are

$$\frac{1}{n} \binom{2n-2}{n-1}$$

plane trees on n vertices, a Catalan number. To prove Theorem 1.2, we sum over all a_j in (5), i.e.,

$$\begin{aligned}
[x^n] P_i &= \sum_{a_1 + \cdots + a_k = n-1} \frac{1}{n} \prod_{j=1}^{k} \binom{n + a_j - 1}{a_j} = \frac{1}{n} \sum_{a_1 + \cdots + a_k = n-1} \prod_{j=1}^{k} \binom{-n}{a_j} (-1)^{a_j} \\
&= \frac{1}{n} \sum_{a_1 + \cdots + a_k = n-1} (-1)^{a_1 + \cdots + a_k} \prod_{j=1}^{k} \binom{-n}{a_j} = \frac{1}{n} (-1)^{n-1} \binom{-kn}{n-1} = \frac{1}{n} \binom{kn + (n-1) - 1}{n-1} \\
&= \frac{1}{n} \binom{(k+1)n - 2}{n-1}.
\end{aligned}$$

Alternative proof of Theorem 1.2. Let $P_i(x)$ be the generating function for weakly labelled k-plane trees where x marks a vertex. Then

$$P_i = x \cdot \frac{1}{1-P_1(x)} \cdot \frac{1}{1-P_2(x)} \cdots \frac{1}{1-P_k(x)}.$$

Since $P_1 = P_2 = \cdots = P_k$, then

$$P_i = x\frac{1}{(1-P_i(x))^k}.$$

By Lagrange inversion formula, we obtain

$$\begin{aligned}[x^n]P_i &= \frac{1}{n}[p^{n-1}](1-p)^{-kn} = \frac{1}{n}[p^{n-1}]\sum_{i\geq 0}\binom{-kn}{i}(-p)^i = \frac{1}{n}[p^{n-1}]\sum_{i\geq 0}\binom{kn+i-1}{i}p^i \\ &= \frac{1}{n}\binom{kn+(n-1)-1}{n-1} = \frac{1}{n}\binom{(k+1)n-2}{n-1}.\end{aligned}$$

□

2.2 Root degree and label of the eldest child of the root

We prove the following result.

Lemma 2.2. *There are*

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1} \tag{6}$$

weakly labelled k-plane trees on n vertices with root of degree r such that all r children of the root are labelled by i.

Proof. We extract the coefficient of x^n in xP_i^r as follows.

$$\begin{aligned}[x^n]xP_i^r = [x^{n-1}]P_i^r &= \frac{r}{n-1}[p^{n-r-1}](1-p)^{-k(n-1)} = \frac{r}{n-1}[p^{n-r-1}]\sum_{j\geq 0}\binom{-k(n-1)}{j}(-p)^j \\ &= \frac{r}{n-1}\binom{k(n-1)+(n-r-1)-1}{n-r-1} = \frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}.\end{aligned}$$

□

Upon setting $k=1$ in (6), we obtain the formula for the number of plane trees with n vertices such that the root is of degree r. Summing over all r in (6), we find that there are

$$\frac{1}{n-1}\binom{(k-1)(n-1)}{n-2}$$

weakly labelled k-trees on n vertices.

Theorem 2.3. *The number of weakly labelled k-plane trees on n vertices such that the root is of degree r and the eldest child of the root is labelled i is given by*

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}\binom{k-i+r-1}{r-1}. \tag{7}$$

Proof. By Lemma 2.2, there are

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}$$

weakly labelled k-plane trees on n vertices such that all the r children of the root are labelled j. If the eldest child of the root is to be relabelled i then the siblings of the eldest child of the root must be relabelled as i, followed by a sequence of siblings labelled $i+1$, so on until a sequence of siblings labelled k, on the far right. This is equivalent to choosing $r-1$ objects from a set with $k-i+1$ elements such that repetitions are allowed. The number of such ways is

$$\binom{k-i+r-1}{r-1}.$$

The result thus follows by the product rule of counting. □

We obtain the following corollary upon summing over all i in (7), making use of hockey stick identity.

Corollary 2.4. *There are*

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}\binom{k+r-1}{r} \tag{8}$$

weakly labelled k-plane trees on n vertices with root degree r.

The following version of Vandermonde type convolution was obtained in [9, Corollary 5.5].

Identity 2.5. *Let m_1 and m_2 be positive integers and n a nonnegative integer then,*

$$\sum_{j=0}^{n}\binom{m_1+j-1}{j}\binom{m_2+n-j-1}{n-j}=\binom{m_1+m_2+n-1}{n}.$$

If we sum over all r in (8), making use of Identity 2.5, we obtain (1) as the number of weakly labelled k-plane trees on n vertices. Setting $i=k-i+1$ in (7) we obtain:

Corollary 2.6. *The number of weakly labelled k-plane trees on n vertices whose root is of degree r such that the youngest child of the root is labelled i is given by*

$$\frac{r}{n-1}\binom{(k+1)(n-1)-r-1}{n-r-1}\binom{i+r-2}{r-1}. \tag{9}$$

Summing over all r in (7), again making use of Identity (2.5), we arrive at the following result.

Corollary 2.7. *The number of weakly labelled k-plane trees on n vertices in which the eldest child of the root is labelled i is*

$$\frac{2k-i+1}{kn-i+1}\binom{(k+1)n-i-2}{n-2}. \tag{10}$$

By setting $i=1$ in (10), we find that:

Corollary 2.8. *There are*

$$\frac{2}{n}\binom{(k+1)n-3}{n-2} \tag{11}$$

weakly labelled k-plane trees on n vertices in which the eldest child of the root is labelled 1.

When we delete an edge connecting the root vertex to its eldest child and introduce a new vertex as a root so that the eldest child of the initial root becomes the eldest child of the new root and the initial root (now labelled 1) becomes the second child (youngest child) of the new root then, Corollary 2.8 follows easily by setting $n=n+1$ and $r=2$ in (6). The same arguments can be used to show that (11) also counts weakly labelled k-plane trees on n vertices such that the youngest child of the root is labelled k.

Corollary 2.9. *The number of weakly labelled k-plane trees on n vertices in which the youngest child of the root is labelled i is*

$$\frac{k+i}{k(n-1)+i}\binom{(k+1)(n-1)+i-2}{n-2}. \tag{12}$$

Proof. Sum over all r in (9). □

Setting $i=k$ in (12), we find that (10) also gives the number of weakly labelled k-plane trees on n vertices such that the youngest child of the root is labelled k.

2.3 Leaves

We enumerate weakly labelled k-plane trees according to number of leaves.

Theorem 2.10. *There are*

$$\frac{1}{n}\binom{n}{\ell}\sum_{q\geq 0}\binom{n-\ell}{q}(-1)^q\prod_{j=1}^{k}\binom{n-\ell-q+a_j-1}{a_j} \tag{13}$$

weakly labelled k-plane trees on n vertices with ℓ leaves such that a_j vertices are labelled j.

Proof. Let $P_i(x, v, u_1, \ldots, v_k)$ be the multivariate generating function for weakly labelled k-plane trees with roots labelled i where x marks a vertex, v marks a leaf and u_j marks a vertex labelled j for $j = 1, 2, \ldots, k$. Then we have,

$$P_i(x,v) = xv - x + x\cdot\frac{1}{1-u_1P_1(x)}\cdot\frac{1}{1-u_2P_2(x)}\cdots\frac{1}{1-u_kP_k(x)}.$$

Since $P_1 = P_2 = \cdots = P_k$, it follows that

$$P_i = xv - x + x\frac{1}{\prod_{j=1}^{k}(1-u_jP_i)} = x\left(v + \frac{1-\prod_{j=1}^{k}(1-u_jP_i)}{\prod_{j=1}^{k}(1-u_jP_i)}\right).$$

Application of Lagrange inversion formula gives,

$$\begin{aligned}
[x^n v^\ell u_1^{a_1}\cdots u_k^{a_k}]P_i &= \frac{1}{n}[p^{n-1}v^\ell u_1^{a_1}\cdots u_k^{a_k}]\left(v + \frac{1-\prod_{j=1}^{k}(1-u_jp)}{\prod_{j=1}^{k}(1-u_jp)}\right)^n\\
&= \frac{1}{n}[p^{n-1}v^\ell u_1^{a_1}\cdots u_k^{a_k}]\sum_{j\geq 0}\binom{n}{j}v^j\left(\frac{1-\prod_{j=1}^{k}(1-u_jp)}{\prod_{j=1}^{k}(1-u_jp)}\right)^{n-j}\\
&= \frac{1}{n}[p^{n-1}u_1^{a_1}\cdots u_k^{a_k}]\binom{n}{\ell}\left(1-\prod_{j=1}^{k}(1-u_jp)\right)^{n-\ell}\prod_{j=1}^{k}(1-u_jp)^{-(n-\ell)}\\
&= \frac{1}{n}[p^{n-1}u_1^{a_1}\cdots u_k^{a_k}]\binom{n}{\ell}\sum_{q\geq 0}\binom{n-\ell}{q}(-1)^q\prod_{j=1}^{k}(1-u_jp)^{q-(n-\ell)}\\
&= \frac{1}{n}[p^{n-1}u_1^{a_1}\cdots u_k^{a_k}]\binom{n}{\ell}\sum_{q\geq 0}\binom{n-\ell}{q}(-1)^q\prod_{j=1}^{k}\sum_{b\geq 0}\binom{n-\ell-q+b-1}{b}p^bu_j^b\\
&= \frac{1}{n}\binom{n}{\ell}\sum_{q\geq 0}\binom{n-\ell}{q}(-1)^q\prod_{j=1}^{k}\binom{n-\ell-q+a_j-1}{a_j}.
\end{aligned}$$

The last equality follows since $u_1 + \cdots + u_k = n-1$. Thus the proof. □

This following corollary can also be obtained by summing over all a_j in (13).

Corollary 2.11. *There are*

$$\frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}\binom{(k+1)n-k\ell-ka-2}{n-1}(-1)^a \tag{14}$$

weakly labelled k-plane trees on n vertices with ℓ leaves.

Proof. Let $P_i(x,v)$ be the bivariate generating function for weakly labelled k-plane trees with roots labelled i where x and v marks a vertex and a leaf respectively. Then we have,

$$P_i(x,v) = xv - x + x \cdot \frac{1}{1-P_1(x)} \cdot \frac{1}{1-P_2(x)} \cdots \frac{1}{1-P_k(x)}.$$

Since $P_1 = P_2 = \cdots = P_k$, then

$$P_i = xv - x + x\frac{1}{(1-P_i)^k} = x\left(v + \frac{1-(1-P_i)^k}{(1-P_i)^k}\right).$$

By Lagrange inversion formula, we get

$$\begin{aligned}
[x^n v^\ell]P_i &= \frac{1}{n}[p^{n-1}v^\ell]\left(v + \frac{1-(1-p)^k}{(1-p)^k}\right)^n = \frac{1}{n}[p^{n-1}v^\ell]\sum_{j\geq 0}\binom{n}{j}v^j\left(\frac{1-(1-p)^k}{(1-p)^k}\right)^{n-j} \\
&= \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\left(1-(1-p)^k\right)^{n-\ell}(1-p)^{-k(n-\ell)} \\
&= \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a(1-p)^{ak-k(n-\ell)} \\
&= \frac{1}{n}[p^{n-1}]\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\sum_{b\geq 0}\binom{k(n-\ell-a)+b-1}{b}p^b \\
&= \frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{n-\ell}{a}(-1)^a\binom{(k+1)n-k\ell-ka-2}{n-1}.
\end{aligned}$$

□

Equation (14) can be written as

$$\frac{1}{n}\binom{n}{\ell}\sum_{a\geq 0}\binom{\ell-n+a-1}{a}\binom{(k+1)n-k\ell-ka-2}{kn-k\ell-ka-1}.$$

Setting $k=1$ and summing over all a, we obtain the Nayarana number

$$\frac{1}{n}\binom{n}{\ell}\binom{n-2}{n-\ell-1} = \frac{1}{n-1}\binom{n-1}{\ell}\binom{n-1}{\ell-1}$$

which gives the number of plane trees on n vertices with ℓ leaves.

2.4 Forests

In this subsection, we are interested in obtaining a formula for weakly labelled k-plane forests with a given number of components.

Theorem 2.12. *There are*

$$\frac{c}{n}\binom{(k+1)n-c-1}{n-c}. \tag{15}$$

forests of weakly labelled k-plane trees on n vertices and c components such that the roots of the components are labelled $1,2,\ldots,c$ from left to right.

Proof. Let $P(x) = P$ be the generating function for weakly labelled k-plane trees on n vertices. Then,

$$P = \frac{x}{(1-P)^k}.$$

A forest is a sequence of trees, so we extract the coefficient of x^n in P^c as follows.

$$[x^n]P^c = \frac{c}{n}[p^{n-c}](1-p)^{-kn} = \frac{c}{n}[p^{n-c}]\sum_{j\geq 0}\binom{-kn}{j}(-p)^j = \frac{c}{n}\binom{kn+(n-c)-1}{n-c} = \frac{c}{n}\binom{(k+1)n-c-1}{n-c}.$$

The roots of the components are then labelled $1, 2, \ldots, c$ to avoid redundancies. □

We remark that (15) can be obtained by considering a weakly labelled k-plane tree on $n+1$ vertices such that the root is of degree c and all the children of the root are labelled i. The forest is attained by deleting the root vertex and all the edges incident to it. The roots of the components are then labelled $1, 2, \ldots, c$ from left to right. The formula is thus obtained by setting $n = n+1$ and $r = c$ in (6). Setting $c = 1$ in (15), we obtain the number of weakly labelled k-plane trees on n vertices. Upon setting $k = 1$ in (15), we obtain a formula for forests of plane trees on n vertices and c components.

2.5 Outdegree sequence

Theorem 2.13. *The number of weakly labelled k-plane trees on n vertices such that there are d_i vertices with i children where $i = 0, 1, \ldots$ is*

$$\frac{1}{n}\binom{n}{d_0, d_1, \ldots}\prod_{r\geq 0}\binom{k+r-1}{r}^{d_r}. \tag{16}$$

Proof. There are

$$\frac{1}{n}\binom{n}{d_0, d_1, \ldots}$$

plane trees on n vertices with outdegree sequence $(d_0, d_1, \ldots)$ [14]. By the proof of Theorem 2.3, there are $\binom{r+k-1}{r}$ ways to assign labels to r children of a vertex. By the product rule of counting the result follows. □

We obtain the following identity by summing over all d_i in (16) and using (4).

Identity 2.14. *Let $d_0, d_1, \ldots$ be non-negative integers. Also let k and n be positive integers. Then*

$$\sum_{d_0+d_1+\cdots=n}\binom{n}{d_0, d_1, \ldots}\prod_{r\geq 0}\binom{k+r-1}{r}^{d_r} = \binom{(k+1)n-2}{n-1}.$$

3 Bijections

This section is devoted to construction of bijections of various combinatorial structures which are related to weakly labelled k-plane trees.

3.1 k-ary Husimi graphs

In the sequel, we relate weakly labelled k-plane trees to k-ary Husimi graphs.

Theorem 3.1. *There is a bijection between the set of k-ary Husimi graphs on n vertices and the set of weakly labelled k-plane trees on n vertices.*

Proof. Consider a k-ary Husimi graph on n vertices. We obtain a weakly labelled k-plane tree with n vertices using the following procedure:

(i) By Breadth First Search (BFS), traverse the k-ary Husimi graph and give label i to all the vertices on the block (which appear at a lower level) if the block is the i^{th} block child of its parent. A block child j of a vertex i is a block attached to i such that all other vertices of the block are at a lower level.

(ii) Let $x, x_1, x_2, \ldots, x_m$ be the vertices of a block of size m where x is the root and x_i is a vertex which appears on the left of x_{i+1} for all $i = 1, \ldots, m-1$. Delete all edges connecting x_i's and create edges between x and all x_i's where $i = 1, \ldots, m$. Deletion of edges and creation of new edges ensure that the resultant structure is a plane tree.

(iii) Repeat steps (i) and (ii) until all the vertices of the graph are labelled. Note that no vertex is deleted nor introduced, so the tree obtained has the same number of vertices as the Husimi graph.

Since the labels of all the vertices sharing a parent are dependent on their block child position in the k-ary Husimi graph, then these labels are strictly non-decreasing from left to right, creating a weakly labelled k-plane tree on n vertices. We obtain the reverse procedure as follows:

(i) Starting with a weakly labelled k-plane tree, create a block consisting of vertices on the same level but with the same label, together with their parent.

(iii) Position the block children according to the labels on the vertices and delete the labels, i.e, if the vertices are labelled i, then the block takes position i in the k-ary Husimi graph. The structure obtained is a k-ary Husimi graph.

Figure 2 is a depiction of the bijection. □

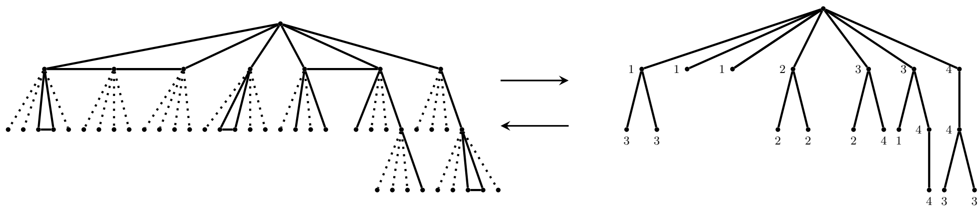

Figure 2: A 4-ary Husimi graph on 20 vertices with its corresponding weakly labelled 4-plane tree on 20 vertices.

3.2 Plane trees with given number of leaves

We prove the following result:

Theorem 3.2. *There is a bijection between the set of weakly labelled k-plane trees on n vertices and the set of plane trees with kn vertices where each internal vertex has exactly $k-1$ leaves.*

Proof. We construct a plane tree satisfying the stated conditions from a given a weakly labelled k-plane tree on n vertices by the following procedure: For each vertex of the weakly labelled k-plane tree, if there is no child of the vertex labelled $1 \leq i \leq k$, draw an imaginary leaf labelled i. Now, draw a leaf separating vertices of labels j and $j+1$ where $1 \leq j \leq (k-1)$. Delete the imaginary leaves and the labels of the vertex and its children. Each internal vertex (inclusive of the root) has $k-1$ leaves, since the drawn leaves separate vertices of labels j and $j+1$ where $1 \leq j \leq (k-1)$. The resulting structure is a plane tree with kn vertices (each internal vertex contributes $k-1$ leaves, giving a total of $n(k-1)$ leaves which together with the initial n internal vertices give a total of kn vertices).
Conversely, we obtain a weakly labelled k-plane tree with n vertices from a plane tree with kn vertices such that each internal vertex has exactly $k-1$ leaves as follows:

(i) Consider all non-leaf children of each vertex x and let $\ell_1, \ell_2, \ldots, \ell_{k-1}$ be the leaf children of x labelled from left to right. Label all non-leaf children of x on the left of ℓ_1 as 1, all non-leaf children between ℓ_i and ℓ_{i+1} as $i+1$ for $i = 2, \ldots, k-2$ and those on the right of ℓ_{k-1} (if any) as k.

(ii) Delete all the leaves.

The resulting structure has all vertices on the same level labelled with non-decreasing values and the vertices have labels from the set $\{1, 2, \dots, k\}$ from left to right hence its a weakly labelled k-plane tree.
See Figure 3 for an example. □

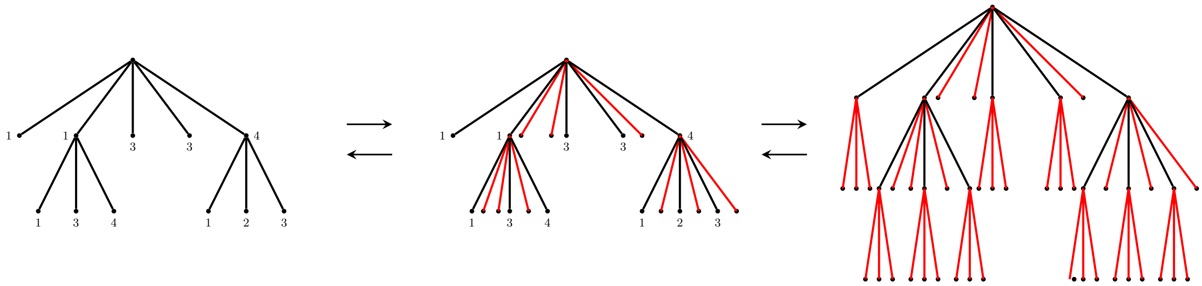

Figure 3: A weakly labelled 4-plane tree on 12 vertices and its corresponding plane tree on 48 vertices where each internal vertex has exactly 3 leaves.

3.3 Weakly labelled k-trees and k-plane trees

In this subsection, we construct two bijections regarding weakly labelled k-plane trees and k-plane trees as well as weakly labelled k-trees and k-plane trees.

Theorem 3.3. *There is a bijection between the set of weakly labelled k-plane trees on n vertices and the set of k-plane trees on n vertices whose roots are labelled* 1.

Proof. Consider a weakly labelled k-plane tree T with n vertices. We obtain its corresponding k-plane tree on n vertices by the following procedure: We traverse T using BFS.

(i) Label the root of the tree as 1.

(ii) Let x be a non-root vertex and $x_1, x_2, \dots, x_m$ be its children arranged in this order from left to right. Also let y be the label of the parent of x. Let the labels of $x, x_1, x_2, \dots, x_m$ be $s, t_1, t_2, \dots, t_m$ respectively. Moreover, let x_r be the first vertex visited with the property that $s + t_r > k + 1$. This means that $s + t_i > k + 1$ for all $i \geq r$. Let S be a subtree of T rooted at x with $x_r, x_{r+1}, \dots, x_m$ as its children. Adjoin S to the parent of x such that $x_r, x_{r+1}, \dots, x_m$ are on the immediate right of x in this order.

(iii) Assign the label $s + t_i - (k + 1)$ to the vertices initially labelled x_i for $i = r, r + 1, \dots, m$.

(iv) Repeat steps (ii) and (iii) until all the vertices are traversed.

Since for each $i = 1, 2, \dots, m$, we have $t_i \leq k$ and $y + s \leq k + 1$ then $y + (s + t_i - (k + 1)) \leq t_i \leq k < k + 1$ which satisfies the property that the sum of the labels of any adjacent vertices is at most $k + 1$. Hence the structure obtained is a k-plane tree.

Conversely, we obtain a weakly labelled k-plane tree from a k-plane tree P, by the following procedure: We traverse P using BFS.

(i) Delete the label of the root.

(ii) Let vertex x be a vertex of P with the property that its label is s and it has at least one sibling on its right with a label less than s. Moreover, let y be the first sibling of x, on its right, with a label greater than or equal to s. Note that we may not have such a vertex. Let $x_1, x_2, \dots, x_m$ be the children of the root of x which appear in this order from left to right between x and y or all of them to the right of x if there is no such a vertex y. Let the labels of $x_1, x_2, \dots, x_m$ be $t_1, t_2, \dots, t_m$ respectively. Detach the subtrees rooted at $x_1, x_2, \dots, x_m$ and attach them, in this order from left to right, to x on the right of the existing children.

(iii) Relabel vertex x_i as $t_i - s + k + 1$ for $i = 1, 2, \ldots, m$.

(iv) Repeat steps (ii) and (iii) until all the vertices are traversed.

Since for each $i = 1, 2, \ldots, m$, we have $t_i < s$, then $t_i - s + k + 1 < k + 1$. So, the vertices are labelled with integers less than $k+1$. Also, $s + (t_i - s + k + 1) \geq k + 1$. So the vertices $x_1, x_2, \ldots, x_m$ are relabelled with integers greater than their initial labels . Hence the structure obtained is a weakly labelled k-plane tree. This bijection is shown in Figure 4. □

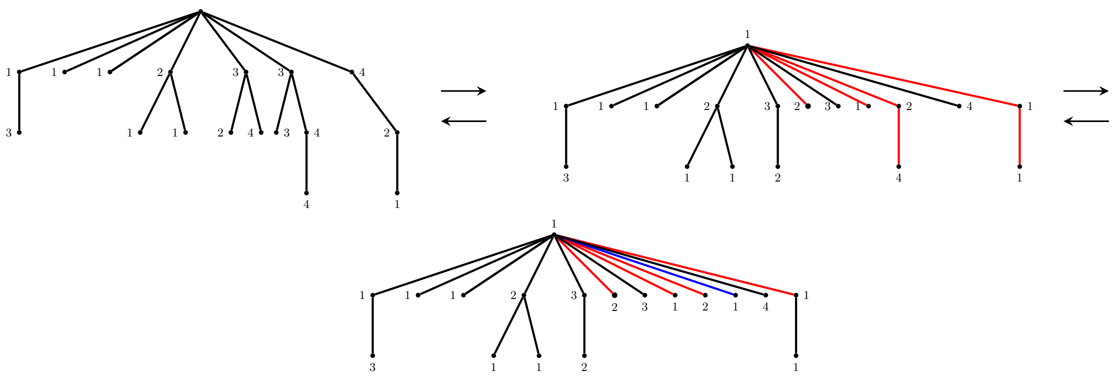

Figure 4: A weakly labelled 4-plane tree and its corresponding 4-plane tree with root labelled 1.

By imitating the procedure used in the proof of Theorem 3.3, we obtain a corresponding k-plane tree given a weakly labelled k-tree. The procedure is easily reversed. The result is stated in the following theorem.

Theorem 3.4. *There is a bijection between the set of k-plane trees with root labelled k on n vertices and the set of weakly labelled k-trees on n vertices.*

The bijection of Theorem 3.4 is with illustrated in the Figure 5.

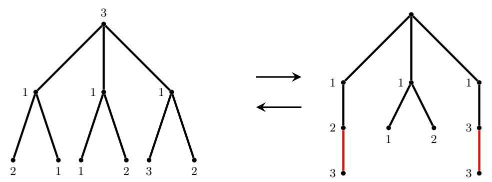

Figure 5: A 3-plane tree with root labelled 3 and its corresponding weakly labelled 3-tree on 10 vertices.

4 Conclusions and future work

In this article, we introduced a new class of k-plane trees and derived their enumerative formulas with respect to various parameters, including number of vertices, occurrences of vertices of a given type, root degree, label of the eldest child of the root, label of the youngest child of the root, number of leaves, associated forests, and outdegree sequence. Furthermore, we established bijections between this family of trees and several well-known combinatorial structures, such as k-ary Husimi graphs, k-plane trees, plane trees in which each internal vertex has a prescribed number of leaf children and weakly labelled k-trees. This work can be extended by establishing bijections between the set of weakly labelled k-plane trees and other combinatorial structures such as lattice paths. Further exploration of weakly labelled k-plane trees under the additional restriction that the sum of the labels of adjacent vertices does not exceed $k + 1$ is recommended.

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